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BULLETIN OF THE UNIVERSITY OF KANSAS

VOL. XV. NOVEMBER 1, 1913. No. 2.

SCIENCE BULLETIN,

Vol. VII, Nos. 1–17.

(Continuation of Kansas University Quarterly.)

LAWRENCE, KANSAS.

Published Semimonthly from January to June and Monthly from July to December, inclusive, by the University of Kansas.

Entered as second-class matter December 29, 1910, at the post office at Lawrence, Kansas, under the act of July 16, 1894.

4-5259

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THE KANSAS UNIVERSITY SCIENCE BULLETIN.

DEVOTED TO THE PUBLICATION OF THE RESULTS OF RESEARCH BY MEMBERS OF THE UNIVERSITY OF KANSAS.

VOL. VII.

PUBLISHED BY THE UNIVERSITY,
LAWRENCE, KANSAS.
1913.

KANSAS STATE PRINTING OFFICE.
W. C. AUSTIN, State Printer.
TOPEKA. 1913.
4 5259

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THE KANSAS UNIVERSITY SCIENCE BULLETIN

VOL. VII, No. 1.] JUNE, 1913. [WHOLE SERIES, VOL. XVII, No. 1.

THE INTERRELATIONS OF EIGHT FUNDAMENTAL PROPERTIES OF CLASSES OF FUNCTIONS.

BY

ARTHUR DUNN PITCHER.

Preface.

In his *Introduction to a Form of General Analysis** MOORE defines General Analysis as "*the theory of classes of functions, functional operations, etc., involving at least one general variable.*" The same memoir contains a theory of classes $\mathfrak{M}$ of real-valued, single-valued functions† μ of a general variable p on a *general range* $\mathfrak{P}$. This theory relates to certain properties of classes $\mathfrak{M}$ of functions.

A property of functions μ or of classes $\mathfrak{M}$ of functions is said to be of *general reference* in case it is defined for functions or classes of functions on a general range $\mathfrak{P}$. Thus *finiteness* on the range $\mathfrak{P}$ and *constancy* on the range $\mathfrak{P}$ are properties of general reference of functions μ. On the other hand a property of functions μ or of classes $\mathfrak{M}$ of functions which refers to some special feature of the range $\mathfrak{P}$ is a property of *special reference*. *Continuity* is a property of special reference of

* Cf. E. H. MOORE, *Introduction to a form of General Analysis, The New Haven Mathematical Colloquium* (Yale University Press, New Haven, 1910), pp. 1–150. In the sequel we refer to this paper as I. G. A. An exposition of the fundamental notions of his theory and its applications is given by the same author in his paper, *On a Form of General Analysis, with Application to Linear Differential and Integral Equations*, read before the Section on Analysis of the Rome Congress of 1908, Atti, etc., Vol. 2 (1909), pp. 98–114.

† In this paper the term function is applied only to real-valued single-valued functions.

functions μ since its definition is relative to some special feature, e. g., *limit* or *distance* postulated as defined for the range $\mathfrak{P}$.

The principal properties of classes of functions used in the above memoir are:

$$(1) \qquad \begin{gathered} L \text{ (linear)};\ C \text{ (closed)};\ D \text{ (dominance)};\ A \text{ (absolute)}; \\ \Delta;\ K_1;\ K_2;\ K_{12*};\ K_{1*};\ K_{2*};\ D_1. \end{gathered}$$

These properties are fully defined in the introductory sections of this paper. The first eight are the more important. The first four and the last are properties of general reference. The others are properties of special reference being defined relative to a so-called development Δ of the class $\mathfrak{P}$. Moore shows that these properties belong to many classes of functions of well-known importance in Analysis. Thus the classes: all numbers; all n-partite numbers; all absolutely convergent series; all sequences converging to 0; all continuous functions on a closed linear interval, each possess the first eight properties.

In the theory of Moore the development Δ of the range $\mathfrak{P}$ is of fundamental importance and, as a basis of definition of properties of infinitary or infinitesimal nature like the properties K_1, K_2 here involved,* is no doubt destined to play an important rôle in general function theory. In this paper various types of developments of $\mathfrak{P}$ singular ($\mathfrak{P}^{\mathrm{I}}$), $\mathfrak{P}$ dual ($\mathfrak{P}^{\mathrm{II}_2}$), $\mathfrak{P}$ triple ($\mathfrak{P}^{\mathrm{II}_3}$), $\mathfrak{P}$ finite ($\mathfrak{P}^{\mathrm{II}_n}$), $\mathfrak{P}$ denumerable ($\mathfrak{P}^{\mathrm{III}}$) and $\mathfrak{P}$ general are specified.† All of the independence examples and many of the theorems are in terms of these special developments.

In the Introduction to a Form of General Analysis the importance of the property K_{12*} is displayed and indeed it is one of the goals of the theory. The possession of this property by a class $\mathfrak{M}'$ of functions insures a certain functional characterization of the so-called $*$-composite class,‡ $(\mathfrak{M}'\mathfrak{M}'')_*$, where $\mathfrak{M}''$ is an arbitrary class of functions having the properties L, C, D. Such a characterization is of fundamental importance in the development of the general theory of differential and integral equations from the standpoint of General

* Cf. I. G. A., § 67.

† §§ 15; 16; 47.

‡ E. g., if $\mathfrak{M}'$ and $\mathfrak{M}''$ are, respectively, the classes of all continuous functions of p' and p'' where p' and p'' have the respective ranges, $0 \leqq p' \leqq 1$, $0 \leqq p'' \leqq 1$; then the $*$-composite class $(\mathfrak{M}'\mathfrak{M}'')_*$ is the class of all continuous functions of the bipartite variable $p'p''$ which ranges over the plane region: $0 \leqq p' \leqq 1$, $0 \leqq p'' \leqq 1$.

Analysis. Moore shows that the property K_{12*} is implied by the properties D; Δ; K_1; K_2 and that the genus of systems:

$$\Sigma = (\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M}),^*$$

whose class $\mathfrak{M}$ has the properties:

$$D; \Delta; K_1; K_2,$$

is closed under composition of classes and developments.† In §§33; 35; 40 we indicate three other extensive genera of systems:

$$\Sigma = (\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M}),$$

which are similarly closed under composition and whose classes $\mathfrak{M}$ have this fundamental property K_{12*}.

This paper had its origin in an attempt to develop the *complete existential theory*, soon to be defined, of the properties:

$$L; C; D; A; \Delta; K_1; K_2; K_{12*}. \tag{2}$$

This study has proved to be extremely interesting and suggestive and has given rise to many theorems not immediately connected with the complete existential theory. Such an investigation in connection with the fundamental postulates of other mathematical disciplines would no doubt prove to be equally fruitful. In our investigations we have made free use of the properties:

$$D_1; K_{1*}; K_{2*}, \tag{3}$$

and many of the theorems involve these properties.

A property P, defined for the elements of a certain class or for systems of a certain type, is said to be *existent* or *non-existent* (for that class or type) in case there exists or does not exist an element of the class in question or a system of the type in question possessing the property P.

Consider in general n properties:

$$P_1; P_2; \cdots; P_n, \tag{4}$$

and their negatives:

$$^-P_1; ^-P_2; \cdots; ^-P_n, \tag{5}$$

of systems Σ of a certain type. Consider also the 2^n composite

* A system $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$ is a class $\mathfrak{P}$ of elements with a development Δ, and a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$. ($\mathfrak{A}$ denotes the class of all real numbers.)

† I. G. A., § 84.

properties:

$$(6) \quad (++\cdots++);\ (++\cdots+-);\ \cdots(--\cdots-+);\ (--\cdots--),$$

formed from the properties (4) and (5), the part i of each composite property being $+$ or $-$ according as the property P_i is taken positively or negatively. The *complete existential theory** of the n properties (4) of systems Σ consists of a body of 2^n propositions, one relative to each of the composite properties (6). Each proposition states that for the required type of systems Σ, the composite property in question is existent or non-existent. If the proposition is one of existence it is proved by the exhibition of a suitable system Σ. If the proposition is one of non-existence it is proved by a general argumentation showing a relation amongst the properties (4).

The n properties (4) are said to be *completely independent* (*and mutually consistent*) in case the 2^n propositions of the complete theory are propositions of existence. The n properties are independent in the usual sense in case there exist n systems Σ each failing to have precisely one of the n properties so that no one of the n properties is implied by the remaining $n-1$ properties.

We show that of the $2^8 = 256$ composite properties arising from the properties (2), 145 are existent and 108 non-existent. We are unable to decide whether the three composite properties:

$$(7) \quad LC^-DA\Delta K_1K_2^-K_{12*};\ L^-C^-DA\Delta K_1K_2^-K_{12*};\ LC^-D^-A\Delta K_1K_2^-K_{12*},$$

are existent or non-existent, but we do show that for certain very extensive regions of the general field they are non-existent.

Each of the 145 independence examples consists of a particular system $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$. An exhaustive study is made of such systems in the various cases where $\mathfrak{P}$ is $\mathfrak{P}^{\mathrm{I}}$; $\mathfrak{P}^{\mathrm{II}_2}$; $\mathfrak{P}^{\mathrm{II}_3}$; $\mathfrak{P}^{\mathrm{II}_n}$ and a complete theory is given for each of these cases. The known 145 existent composite properties are all existent for systems $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$ where $\mathfrak{P}$ is denumerable.

For the convenience of the reader we give, in the introductory sections, an exposition of those fundamental notions of the Introduction to General Analysis which are essential to the comprehension of this paper, together with those propositions and theorems frequently

* I. G. A., § 47.

used in the sequel. We do not hesitate to use the exact language of the above memoir when it seems best to do so for our purpose. For convenience we include in the introduction a section on special types of developments Δ.

We follow Moore in the use of certain logical symbols which are extremely convenient and almost necessary. In practice these symbols mightily aid in argumentation by the complete and explicit representation of the notions in thoroughly convenient and concise form. We begin with a list of these symbols, and in the introductory sections we translate sufficiently many of the symbolical statements so that throughout the paper the reader should find no difficulty in understanding the notations.

It is with pleasure that, in concluding this preface, we express our obligation to Professor Moore for aid and inspiration freely given in the preparation of this paper.

Introduction.—An exposition of certain of the fundamental notions of the General Analysis of Moore. The definition of certain special types of developments Δ of classes $\mathfrak{P}$. §§ 1–23.

1. *List of logical symbols with interpretations.**—

$=$ logical identity

$\neq$ logical diversity

$\equiv$ definitional identity

$.\supset.$ (for every $\cdots$) it is true that
( ) implies ( )
if ( ) then ( )

$.\subset.$ ( ) is implied by ( )

$.\infty.$ ( ) is equivalent to ( )
( ) implies and is implied by ( )

$\supset$; $\subset$; ∞ implies; is implied by; is equivalent to (as relations of properties)

$\exists$ there exists a (system; class; element; etc.)

$\ni$ such that; where

$\cdot$ and

..; : : ; .: :. signs of punctuation in connection with signs of implication, etc.; the principal implication of a sentence has its sign accompanied with the largest number of punctuation dots.

* I. G. A., pp. 150.

$\smile$ or
$-$ not
[ ] a class of (elements; functions; etc.)
[all] the class consisting of all (elements; functions; etc.; having a specified property or satisfying a specified condition)
$\cup[\mathfrak{P}]$ the least common superclass of the classes $\mathfrak{P}$ of the class $[\mathfrak{P}]$ of classes
$\cap[\mathfrak{P}]$ the greatest common subclass of the classes $\mathfrak{P}$ of the class $[\mathfrak{P}]$ of classes

2. The General Analysis of Moore is concerned with the theory of *real-valued, single-valued functions* on a *general variable.* Otherwise stated it is concerned with functions on $\mathfrak{P}$ to $\mathfrak{A}$ where $\mathfrak{P} \equiv [p]$ is a *general class of elements** and $\mathfrak{A} \equiv [a]$ is the *class of all real numbers.* A function assigns to each element p a real number a.

Moore adopts the terminology of the theory of functions.† Denote a function on $\mathfrak{P}$ to $\mathfrak{A}$ by μ. Then p is the *variable* of the function μ; the class $\mathfrak{P}$ is the *range* (of the variable) of the function μ; at the *argument value* or *place* p the function μ has the *functional value* μ_p, etc.

Such a function μ is indicated by the various notations:

$$\mu, \quad (\mu_p), \quad (\mu_p|p), \quad (\mu_p|p^{\mathfrak{p}}).$$

The last notation may be read: " The system of (functional) values μ_p where p varies over the class $\mathfrak{P}$." *The intention is to discriminate sharply between function and functional value.*

The general theory relates to properties of classes $\mathfrak{M}$ of systems $(\mathfrak{A}; \mathfrak{P}; \mathfrak{M})$ where $\mathfrak{A}$ and $\mathfrak{P}$ are defined as above and

$$\mathfrak{M} \equiv [\mu]$$

is a class of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

In particular the following special systems‡ are noticed:

(I) $\mathfrak{P}^{\mathrm{I}}$ is the class of a single element: e. g., $p \equiv 1$.
$\mathfrak{M}^{\mathrm{I}}$ is the class of all functions on $\mathfrak{P}^{\mathrm{I}}$ to $\mathfrak{A}$, i. e., in effect $\mathfrak{M}^{\mathrm{I}}$ is the real number system $\mathfrak{A}$.

(II_n) $\mathfrak{P}^{\mathrm{II}_n}$ is the class of n elements: e. g., $\mathfrak{P} \equiv 1, 2, \cdots n$.

* German capitals are used to denote classes. The elements of these classes are denoted by the corresponding small Greek or Latin letters.

† I. G. A., § 4. Moore here indicates a more general function F on $\mathfrak{P}$ to $\mathfrak{P}'$.

‡ I. G. A., § 5.

$\mathfrak{M}^{\mathrm{II}_n}$ is the class of all functions μ on $\mathfrak{P}^{\mathrm{II}_n}$ to $\mathfrak{A}$, viz., of all n-partite real numbers:

$$\mu \equiv (\mu_p|p) \equiv (\mu_1, \mu_2, \cdots, \mu_n).$$

(III) $\mathfrak{P}^{\mathrm{III}}$ is the class of a denumerable infinitude of elements: e. g.,

$$p \equiv 1, 2, 3, \cdots, n, \cdots.$$

$\mathfrak{M}^{\mathrm{III}}$ is the class of all functions μ on $\mathfrak{P}^{\mathrm{III}}$ to $\mathfrak{A}$, viz., of all infinite sequences of real numbers:

$$\mu \equiv (\mu_p|p) \equiv (\mu_1, \mu_2, \cdots, \mu_n, \cdots).$$

(III_0) $\mathfrak{M}^{\mathrm{III}_0}$ is the subclass consisting of all functions μ such that

$$\lim_{p=\infty} \mu_p = 0.$$

(III_e) $\mathfrak{M}^{\mathrm{III}_e}$ is the subclass of all functions μ such that

$$\sum_{p=1}^{\infty} A\mu_p^e$$

converges.*

(IV) $\mathfrak{P}^{\mathrm{IV}}$ is the finite interval† (0 1)

$$0 \leqq p \leqq 1$$

of the real system number $\mathfrak{A}$.

$\mathfrak{M}^{\mathrm{IV}}$ is the class of all continuous functions of p on $\mathfrak{P}^{\mathrm{IV}}$ to $\mathfrak{A}$.

3. *Relatively uniform convergence.*—Moore introduces the notion of *relatively uniform convergence.*‡ In order to introduce the notations and clarify the idea we indicate here the various kinds of convergence reserving until last the relatively uniform convergence.§

The sequence $\{\mu_n\}$ of functions μ_n on $\mathfrak{P}$ to $\mathfrak{A}$ converges to the function θ as limit:

(1) $$\lim_{n} \mu_n = \theta,$$

in case

(2) $$p \,.\, e : \supset : \exists\, n_{pe} \,.\, > n_{pe} \,.\, \supset \,.\, A(\mu_{np} - \theta_p) \leqq e,$$

viz., for every p and e‖ (it is true that) there exists a positive integer

* A denotes absolute value of.

† The theory relates equally to case IV for any interval $(a_0 a_1)$.

‡ I. G. A., § 7. This is an instance of relative uniformity in general. See I. G. A., § 6.

§ I. G. A., § 7.

‖ e denotes a positive number greater than zero. The class of all such is denoted by $[e]$.

n_{pe} (dependent on p and e) such that for every n exceeding n_{pe} (it is true that) the absolute value of $(\mu_{np} - \theta_p)$ is at most e.

The convergence is said to be uniform over the range $\mathfrak{P}$:

$$\lim_n \mu_n = \theta \ (\mathfrak{P}), \tag{3}$$

in case

$$e : \supset : \exists\, n_e \ni n > n_e \,.\supset.\, A(\mu_{np} - \theta_p) \leqq e \ (p),^* \tag{4}$$

viz., for every e there exists a positive integer n_e such that for every n exceeding n_e the absolute of the function $(\mu_n - \theta)$ is for every p at most e.

For a function σ, *the absolute of* σ, in notation: $A\sigma$, is the function on $\mathfrak{P}$ to $\mathfrak{A}$ for which $(A\sigma)_p = A(\sigma_p)$ for every p.

The sequence $\{\mu_n\}$ is said to converge to θ uniformly on the range $\mathfrak{P}$ relative to or as to the scale σ:

$$\lim_n \mu_n = \theta \ (\mathfrak{P}; \sigma), \tag{5}$$

in case

$$e : \supset : \exists\, n_e \ni n > n_e \,.\supset.\, A(\mu_n - \theta) \leqq eA\sigma, \tag{6}$$

that is,

$$e : \supset : \exists\, n_e \ni n > n_e \,.\supset.\, A(\mu_{np} - \theta_p) \leqq eA\sigma_p \ (p), \tag{6'}$$

viz., for every e there exists a positive integer n_e such that for every n exceeding n_e it is true that for every p the absolute value of $(\mu_{np} - \theta_p)$ is at most e multiplied by the absolute value of σ_p.

Uniform convergence is an instance of relatively uniform convergence the scale function being the constant function 1.

We speak of the convergence of the sequence $\{\mu_n\}$ on $\mathfrak{P}$; uniformly on $\mathfrak{P}$; uniformly on $\mathfrak{P}$ as to σ, in case there exists a function θ effective as the limit of the sequence:

$$\lim_n \mu_n = \theta \ (p); \quad \lim_n \mu_n = \theta \ (\mathfrak{P}); \quad \lim_n \mu_n = \theta \ (\mathfrak{P}; \sigma),$$

for the respective modes of convergence.†

3*a*. *Relative uniformity as to a class of functions.*—A sequence $\{\mu_n\}$ of functions is said to converge to a function θ uniformly as to a scale class $\mathfrak{S} = [\sigma]$, in notation:

$$\lim_n \mu_n = \theta \ (\mathfrak{P}; \mathfrak{S}),$$

* The significance of the (p) at the end of a statement is that the previous relations hold for every p of the range $\mathfrak{P}$

† I. G. A., § 8.

in case there exists a function σ of $\mathfrak{S}$ such that the sequence $\{\mu_n\}$ converges to θ uniformly as to σ, that is in case

$$\exists\, \sigma \ni \mathop{\mathrm{L}}_n \mu_n = \theta \ (\mathfrak{P}; \sigma).$$

4. *Definitions.**—In §4*a* are stated the simple relations holding between various cases of relatively uniform convergence, the sequences being the same and the scale functions different.

A function σ is said to be *dominated by*† a function τ in case $A\sigma \leqq A\tau$, that is, in case $A\sigma_p \leqq A\tau_p$ for every p.

A class $\mathfrak{S} \equiv [\sigma]$ of functions is said to be dominated by a class $\mathfrak{T} = [\tau]$ of functions in case for each function σ there exists a function τ depending on σ such that σ is dominated by τ.

4*a*. *Propositions.*‡—(1) Uniformity (of convergence) as to σ implies uniformity as to every function dominating σ.

(2) Uniformity as to σ and uniformity as to $a\sigma$ $(a \neq 0)$ are equivalent.

(3) Uniformity as to 0 implies that the functions of the sequence are ultimately equal and hence implies uniformity as to every function σ.

(4) If $A\sigma$ is uniformly bounded from ∞, viz.,

$$\exists\, e \ni A\sigma \leqq e,$$

then uniformity as to σ implies uniformity (i. e., ordinary uniform convergence). Similarly if $A\sigma$ is uniformly bounded from 0, viz.,

$$\exists\, e \ni e \leqq A\sigma,$$

then uniformity implies uniformity as to σ.

5. *The extension of a class $\mathfrak{M}$ of functions as to a function σ or as to a class $\mathfrak{S}$ of functions.*§—$\mathfrak{M}_\sigma$, *the class $\mathfrak{M}$ extended as to the function σ,* is the class of all functions θ of the form:

$$\theta = \mathop{\mathrm{L}}_n \mu_n \ (\mathfrak{P}; \sigma),$$

viz., of all limit functions of sequences $\{\mu_n\}$ of the class $\mathfrak{M}$ converging on $\mathfrak{P}$ uniformly as to the function σ. In symbols:

$$\mathfrak{M}_\sigma = [\text{all } \theta = \mathop{\mathrm{L}}_n \mu_n \ (\mathfrak{P}; \sigma)].$$

* I. G. A., §§ 9; 18.

† This property dominated by is a relation between the two functions σ and τ. In practice it is thought of as a property, in notation: $B_1\tau$, of the function σ. Similarly in case the class $\mathfrak{S}$ is dominated by the class $\mathfrak{T}$, the class $\mathfrak{S}$ is said to have the property $B_1\mathfrak{T}$.

‡ I. G. A., § 9*a*.

§ I. G. A., § 14

$\mathfrak{M}_{\mathfrak{S}}$, *the class* $\mathfrak{M}$ *extended as to the class* $\mathfrak{S}$, is the class of all functions θ of the form:

$$\theta = \mathop{\mathbf{L}}_{n} \mu_n \ (P; \sigma),$$

viz., of all limit functions of sequences $\{\mu_n\}$ of the class $\mathfrak{M}$ converging on $\mathfrak{P}$ uniformly as to some function σ of $\mathfrak{S}$. In symbols:

$$\mathfrak{M}_{\mathfrak{S}} = [\text{all } \theta = \mathop{\mathbf{L}}_{n} \mu_n \ (\mathfrak{P}; \sigma)].*$$

6. *Closure properties.*—In the theory of Moore certain closure properties of classes of functions are fundamental. Of these closure properties we will be principally concerned with

$$A \text{ (absolute)}; \ L \text{ (linear)}; \ C \text{ (closed)}.$$

Classes possessing these properties are denoted by

$$\mathfrak{M}^A; \ \mathfrak{M}^L; \ \mathfrak{M}^C,$$

respectively.

(1) $$\mathfrak{M}^A :\equiv : \mathfrak{M} \ni \mu \ . \supset . \ (A\mu)^{B_0\mathfrak{M}},\dagger$$

viz., the class $\mathfrak{M}$ is absolute in case $\mathfrak{M}$ is such that for every μ (it is true that) the function $A\mu$ belongs to $\mathfrak{M}$. Thus the class of all continuous functions is absolute since the absolute of a continuous function is a continuous function.

(2) $$\mathfrak{M}^L :\equiv : \mathfrak{M} \ni \mu_1 \ . \ \mu_2 \ . \ a_1 \ . \ a_2 \ . \supset . \ (a_1\mu_1 + a_2\mu_2)^{B_0\mathfrak{M}},$$

viz., the class $\mathfrak{M}$ is linear in case for any μ_1 and μ_2 of $\mathfrak{M}$ and any real numbers a_1 and a_2, the function $a_1\mu_1 + a_2\mu_2$ belongs to $\mathfrak{M}$. Thus the class of all convergent series is linear.

(3) $$\mathfrak{M}^C . \equiv . \ \mathfrak{M} \ni \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M},$$

viz., $\mathfrak{M}$ is closed in case $\mathfrak{M}$ is such that the extension of $\mathfrak{M}$ as to itself

* Here the form of θ is the same as in the definition of $\mathfrak{M}_\sigma$. In the former case, however, the definition is relative to $\mathfrak{M}$ and σ while here the definition is relative to $\mathfrak{M}$ and $\mathfrak{S} \equiv [\sigma]$ and accordingly the two forms have different significations. In the latter case the σ is a part of the arbitrariness inherent in the notion *form* just as is the sequence $\{\mu_n\}$ in both cases.

† Relative to elements and classes in general an element x belonging to a class $\mathfrak{P}$ is denoted by the notations: $x^{\mathfrak{P}}$; $x^{B_0\mathfrak{P}}$, so that the notations p; $p^{\mathfrak{P}}$; $p^{B_0\mathfrak{P}}$ are interchangeable. Similarly a class $\mathfrak{X}$ of elements belonging to a class $\mathfrak{P}$ is denoted by: $\mathfrak{X}^{\mathfrak{P}}$; $\mathfrak{X}^{B_0\mathfrak{P}}$. Belonging to $\mathfrak{P}$ is a relation between an element x or a class $\mathfrak{X}$ of elements and the class $\mathfrak{P}$; but in practice it is convenient to think of it as a property of the element x or class $\mathfrak{X}$ of elements. See footnote to § 4.

is the same as $\mathfrak{M}$, i. e., $\mathfrak{M}$ contains the limit functions of all sequences $\{\mu_n\}$ of $\mathfrak{M}$ which converge uniformly as to some function μ of $\mathfrak{M}$.

We note two other closure properties and a proposition.

(4) $$\mathfrak{M}^{\text{multiplicative as to } \mathfrak{A}} . \equiv : \mathfrak{M} \ni \mu \,.\, a \,.\supset.\, (a\mu)^{B_0\mathfrak{M}}.$$

(5) $$\mathfrak{M}^{\text{additive}} . \equiv : \mathfrak{M} \ni \mu_1 \,.\, \mu_2 \,.\supset.\, (\mu_1 + \mu_2)^{B_0\mathfrak{M}}.$$

(6) $$\mathfrak{M}^{L} \,.\sim.\, \mathfrak{M}^{\text{additive . multiplicative as to } \mathfrak{A}},$$

viz., that the class $\mathfrak{M}$ is linear is equivalent to (implies and is implied by) the class $\mathfrak{M}$ is additive and multiplicative as to constants.

7. *Dominance properties.**—The *dominance properties* of classes of functions with which we shall be concerned are D, D_1.

A class $\mathfrak{M}$ of functions has the *dominance property* D in case for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there exists a function μ of $\mathfrak{M}$ and a sequence $\{a_n\}$ of real numbers such that for every n the function μ_n is dominated by the function $a_n\mu$. In symbols:

$$\mathfrak{M}^{D} \,.\equiv.\, \mathfrak{M} \ni \big(\{\mu_n\} \,.\supset.\, \exists\ (a_n \,.\, \mu) \ni A\mu_{np} \leqq Aa_n\mu_p\ \ (np)\big).$$

A class $\mathfrak{M}$ of functions is said to have the *dominance property* D_1 in case for every two functions μ_1 and μ_2 of $\mathfrak{M}$ there exists a number a and a function μ of $\mathfrak{M}$ such that the function $a\mu$ dominates μ_1 and μ_2. In symbols:

$$\mathfrak{M}^{D_1} \,.\equiv.\, \mathfrak{M} \ni \big(\mu_1 \,.\, \mu_2 \,.\supset.\, \exists\ (a \,.\, \mu) \ni A\mu_1 \leqq Aa\mu \,.\, A\mu_2 \leqq Aa\mu\big).$$

The following propositions are holding:

(1) $\mathfrak{M}^{D} \,.\supset.\, \mathfrak{M}^{D_1}$; (2) $\mathfrak{M}^{AL} \,.\supset.\, \mathfrak{M}^{D_1}$.

The dominance properties are fundamental and especial importance attaches to the dominance property D. This dominance property enters the theory chiefly as a property of classes functioning as scales of uniformity. The reader is referred to the Introduction to General Analysis, §§21–26, for discussion of the various dominance properties. We state Theorem I of §25 of the above-mentioned work.

THEOREM I. *If a class $\mathfrak{S}^{D}$ is the scale class of each of a sequence of instances of relatively uniform convergence, then the corresponding sequence $\{\sigma_n\}$ of scale functions may be replaced by a single scale function σ effective for each of the instances.*

A similar theorem may be stated for D_1.

* I. G. A., §§ 19; 22.

THEOREM II. *If a class* $\mathfrak{S}^{D_1}$ *is the scale class of each of a finite number of instances of relative uniform convergence, then the corresponding finite sequence* $\{\sigma_n\}$ *of scale functions may be replaced by a single scale function* σ *effective for each of the instances.*

8. *Five properties of general reference.*—The properties A; L; C; D; D_1 defined in the previous sections are properties of general reference* and are possessed by many classes of functions of recognized importance in Analysis. In particular the classes,†

$$\mathfrak{M}^{\mathrm{I}};\ \mathfrak{M}^{\mathrm{II}_n};\ \mathfrak{M}^{\mathrm{III}};\ \mathfrak{M}^{\mathrm{III}_0};\ \mathfrak{M}^{\mathrm{III}_c};\ \mathfrak{M}^{\mathrm{IV}},$$

possess these properties.

9. *The linear and* $*$*-extension of a class* $\mathfrak{M}$ *of functions.*‡—The *linear extension* of the class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$, denoted by

$$\mathfrak{M}_L,$$

is defined to be the class of all functions

$$\mu_L \equiv \sum_{i=1}^{n} a_i \mu_i,$$

where n is any positive integer.

The $*$*-extension* of the class $\mathfrak{M}$, denoted by

$$\mathfrak{M}_*,$$

is by definition the class of functions $\mathfrak{M}_L$ extended as to the class $\mathfrak{M}$. That is,

$$\mathfrak{M}_* \,.\equiv.\, (\mathfrak{M}_L)_{\mathfrak{M}}.$$

9*a*. *Propositions.*—In the following propositions $\mathfrak{N}$ denotes the classes $\mathfrak{M}_L$, $\mathfrak{M}_*$ derived from $\mathfrak{M}$, and $\mathfrak{A}\mathfrak{M} \equiv [\text{all } a\mu]$.

(1)§ $\qquad \mathfrak{M}^{B_0\mathfrak{N}}; \qquad \mathfrak{N}^{B_0M_*}.$

* See preface.

† § 2; I. G. A., §§ 16; 23.

‡ I. G. A., §§ 27; 43. The property L is a property *extensionally attainable.* In an excursus (I. G. A., §§ 28–42), Moore discusses such properties in the theory of classes in general.

§(1) The class $\mathfrak{M}$ of functions belongs to the class $\mathfrak{N}$ of functions and the class $\mathfrak{N}$ belongs to the class $\mathfrak{M}_*$.

(2) For any class $\mathfrak{M}$ the linear extension of $\mathfrak{M}$ is linear.

(3) If a class $\mathfrak{M}$ is linear the linear extension of $\mathfrak{M}$ is the same as $\mathfrak{M}$ and the $*$-extension of $\mathfrak{M}$ is the same as the extension of $\mathfrak{M}$ as to itself.

(4) If a class $\mathfrak{M}$ is linear and closed then the $*$-extension of $\mathfrak{M}$ is the same as $\mathfrak{M}$.

(5) If $\mathfrak{M}$ has the property D_1 then $\mathfrak{N}$ has the property D_1 and is dominated by the class $\mathfrak{A}\mathfrak{M}$ of functions, and the class $\mathfrak{M}_*$ is linear and has the property D_1.

(2) $\mathfrak{M} \,.\supset.\, \mathfrak{M}_L{}^L.$

(3) $\mathfrak{M}^L \,.\supset.\, \mathfrak{M}_L = \mathfrak{M} \,.\, \mathfrak{M}_* = \mathfrak{M}_{\mathfrak{M}}.$

(4) $\mathfrak{M}^{LC} \,.\supset.\, \mathfrak{M}_* = \mathfrak{M}.$

(5) $\mathfrak{M}^{D_1} \,.\supset.\, \mathfrak{N}^{D_1 B_1 \mathfrak{N}^{\mathfrak{M}}} .\, \mathfrak{M}_*{}^{LD_1}.$

(6) $\mathfrak{M}^D \,.\supset.\, \mathfrak{N}^D \,.\, \mathfrak{M}_*{}^{LCD} \,.\, \mathfrak{M}_{**} = \mathfrak{M}_*.$

10. *Composition of classes of elements.**—From two classes

$$\mathfrak{P}' \equiv [p'], \qquad \mathfrak{P}'' \equiv [p'']$$

of elements we derive the *product class*

$$\mathfrak{P}'\mathfrak{P}'' \equiv [(p', p'')] \equiv [p'p''],$$

i. e., the class $\mathfrak{P} \equiv [p]$ whose elements $p \equiv (p', p'')$ are bipartite, the first part p' ranging over $\mathfrak{P}'$ and the second part p'' ranging over $\mathfrak{P}''$. In practice the notation (p', p'') is replaced by $p'p''$.

The classes $\mathfrak{P}'$, $\mathfrak{P}''$ may be the same. We write

$$\mathfrak{P}\mathfrak{P} \equiv [(p_1, p_2)] \equiv [p_1 p_2].$$

Thus p', p'' are generic elements of the respective classes $\mathfrak{P}'$, $\mathfrak{P}''$ conceptually distinct, but not necessarily actually distinct, while p_1, p_2 are independent (conceptually distinct) generic elements of the class $\mathfrak{P}$.

10*a*. *Remark.*†—It should be noticed that the theory of functions and classes of functions on $\mathfrak{P}$ to $\mathfrak{A}$ is applicable to functions and classes of functions on $\mathfrak{P}'\mathfrak{P}''$, etc., to $\mathfrak{A}$. The class $\mathfrak{P}$ in its generality includes classes $\mathfrak{P}'\mathfrak{P}''$, etc.

11. *Reduction of classes of elements and of functions.*‡—A *reduction* R of a class $\mathfrak{P}$ of elements is the transformation of $\mathfrak{P}$ into a subclass $\mathfrak{P}_R \equiv [p_R]$ of $\mathfrak{P}$. Any subclass $\mathfrak{P}_1$ of $\mathfrak{P}$ defines a reduction R of $\mathfrak{P}$, the transformation being the transformation of $\mathfrak{P}$ into $\mathfrak{P}_1$. A reduction R transforms a function μ on $\mathfrak{P}$ to $\mathfrak{A}$ into a function μ_R on

(6) If $\mathfrak{M}$ has the property D then $\mathfrak{N}$ has the property D and $\mathfrak{M}_*$ has the properties LCD and the $*$-extension of $\mathfrak{M}_*$ is the same as $\mathfrak{M}_*$.

For a more extensive body of propositions than the above see I. G. A., §§ 22*a*; 26; 44*a*. Proposition (6) above is of fundamental importance in the general theory.

* I. G. A., § 51.

† I. G. A., § 51*a*.

‡ I. G. A., § 52.

$\mathfrak{P}_R$ to $\mathfrak{A}$. Thus

$$\mu \equiv (\mu_p | p^{\mathfrak{P}}); \qquad \mu_R \equiv (\mu_p\ p^{\iota_R}).$$

We speak of the *reduced class* $\mathfrak{P}_R$ and the corresponding *reduced function* μ_R.

In case a property P of functions is possessed by μ_R the function μ is said to possess the property P on the class $\mathfrak{P}_R$, in notation: $\mu^{P(\mathfrak{P}_R)}$. Similarly in case a property P of classes of functions is possessed by a class $\mathfrak{M}_R \equiv [\mu_R]$, the class $\mathfrak{M} \equiv [\mu]$ is said to possess the property $\mathfrak{P}$ on $\mathfrak{P}_R$, in notation: $\mathfrak{M}^{P(\mathfrak{P}_R)}$.

11*a*. *Propositions.**—The following propositions are relative to a class $\mathfrak{P}$, a reduction R applicable to $\mathfrak{P}$, and classes $\mathfrak{M}$ and $\mathfrak{S}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

(1) $(\mathfrak{M}_L)_R \equiv (\mathfrak{M}_R)_L$.

(2)† $\mathfrak{S}^{B_0\mathfrak{M}} . \supset . \mathfrak{S}_R{}^{B_0\mathfrak{M}_R}$.

(3) $\mathfrak{M}^A . \supset . \mathfrak{M}_R{}^A$.

(4) $\mathfrak{M}^L . \supset . \mathfrak{M}_R{}^L$.

(5) $\mathfrak{M}^{D_1} . \supset . \mathfrak{M}_R{}^{D_1}$.

(6)‡ $\mathfrak{M}^D . \supset . \mathfrak{M}_R{}^D$.

12. *Composition of classes of functions.*§—From any two classes:

$$\mathfrak{M}' \equiv [\mu']; \qquad \mathfrak{M}'' \equiv [\mu''],$$

of functions:

$$\mu' \equiv (\mu'_{p'} | p'); \qquad \mu''_{p''} = (\mu''_{p''p'} | p''),$$

on the classes $\mathfrak{P}'$; $\mathfrak{P}''$ respectively to $\mathfrak{A}$, arises by multiplication of constituent functions the *product class:*

$$\mathfrak{M}'\mathfrak{M}'' \equiv [\mu'\mu''],$$

of functions:

$$\mu'\mu'' \equiv (\mu'_{p'}\, \mu''_{p''} \mid p'p''),$$

on the product class $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$.

It is to be noticed that while the notion of composition of classes of functions is very different from that of composition of classes of elements, the same notations are used in each case. This need cause no confusion.

We desire to call attention to the remark of §10*a*. The theory previously given relative to classes of functions $\mathfrak{M}$ on $\mathfrak{P}$ to $\mathfrak{A}$ applies

* I. G. A., § 52*a*.

† The property *belonging to* is an example of a bipartite property of classes of functions invariant under a reduction of classes. I. G. A., § 52*b*.

‡ The properties A, L, D, D_1 are examples of unipartite properties invariant under reduction. I. G. A., § 52*b*.

§ I. G. A., § 53.

equally well to classes of functions on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ and in particular to the composite class $\mathfrak{M}'\mathfrak{M}''$ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$.

12*a*. *Propositions.**—

$$(1)\qquad \mathfrak{M}'^{D_1}\,.\,\mathfrak{M}''^{D_1}\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1}\,.\,(\mathfrak{M}'\mathfrak{M}'')_L{}^{D_1}\,.\,(\mathfrak{M}'\mathfrak{M}'')_*{}^{D_1}.$$

$$(2)\qquad \begin{aligned}\mathfrak{M}'^{D}\,.\,\mathfrak{M}''^{D}\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D}\,.\,(\mathfrak{M}'\mathfrak{M}'')_{**} &= (\mathfrak{M}'\mathfrak{M}'')_* = (\mathfrak{M}'_*\mathfrak{M}''_*)_* \\ &= (\mathfrak{M}'_*\mathfrak{M}'')_* = (\mathfrak{M}'\mathfrak{M}''_*)_*.\end{aligned}$$

$$(3)\qquad \mathfrak{M}'\,.\,\mathfrak{M}''\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')_L = (\mathfrak{M}'_L\mathfrak{M}''_L)_L = (\mathfrak{M}'_L\mathfrak{M}'')_L = (\mathfrak{M}'\mathfrak{M}''_L)_L.$$

13. *Developments* Δ *of classes*† $\mathfrak{P}$.—A *development* Δ *of a class* $\mathfrak{P}$ *of elements* is a sequence $\{\Delta^m\}$ of systems Δ^m of subclasses of $\mathfrak{P}$, each system Δ^m consisting of a finite number, say l_m, of subclasses of $\mathfrak{P}$. The system Δ^m is *stage* m *of the development* Δ. The sequence $\{l_m\}$ of positive integers is the *sequence of the development* Δ.

There is no question of order, but there may be repetitions amongst the l_m subclasses of stage m, and it is to provide for this that we say system Δ^m instead of class Δ^m.

There is introduced *an index* l^m:

$$l^m = 1,\ 2,\ \cdots,\ l_m,$$

of stage m, but the superscript m of the index l^m is omitted, when in conjunction with the index m of the stage, it occurs in the bi-partite index ml. The l_m subclasses of stage m are denoted respectively by the notations:

$$\mathfrak{P}^{ml},\quad [p^{ml}] \qquad\qquad (l = 1,\ 2,\ \cdots,\ l_m).$$

Then stage m of the development Δ is the system:

$$\Delta^m \equiv (\mathfrak{P}^{ml}),$$

and the development Δ is the system:

$$\Delta \equiv ((\mathfrak{P}^{ml})),$$

of subclasses $\mathfrak{P}^{ml}$ of the class $\mathfrak{P}$.

A *representative system:*

$$\mathfrak{R}^{\Delta} \equiv ((r^{ml})),$$

of a development Δ is a system of elements r^{ml} of $\mathfrak{P}$, the element r^{ml}

* Cf. I. G. A., § 54*a*. D and D_1 are examples of unipartite properties invariant under multiplication of classes.

† I. G. A., § 75.

belonging to the class $\mathfrak{P}^{ml}$ of the development Δ. The element r^{ml} is said to represent the class $\mathfrak{P}^{ml}$ and its elements p^{ml}. It is understood however that only the existent classes $\mathfrak{P}^{ml}$ have representative elements r^{ml}.

14. *Various classes derived from a development* Δ.*—For a development Δ *the class* $\mathfrak{P}^m$:

$$\mathfrak{P}^m \equiv [p^m] \equiv \mathsf{U}\Delta^m \equiv \mathsf{U}(\mathfrak{P}^{ml}),$$

is the class of all elements (in notation: p^m) developed in the stage m, the least common superclass of the classes of the stage m; and *the class* $\mathfrak{P}^\Delta$:

$$\mathfrak{P}^\Delta \equiv [p^\Delta] \equiv \mathsf{U}\Delta \equiv \mathsf{U}((\mathfrak{P}^{ml})) \equiv \mathsf{U}(\mathfrak{P}^m),$$

is the class of all elements (in notation: p^Δ) developed in the development Δ, the least common superclass of the classes $\mathfrak{P}^{ml}$ of Δ.

The class $\mathfrak{P}^{-m}$:

$$\mathfrak{P}^{-m} \equiv [p^{-m}] \equiv \mathfrak{P} - \mathfrak{P}^m,$$

is the class of all elements (in notation: p^{-m}) not developed in stage m, and *the class* $\mathfrak{P}_m$:

$$\mathfrak{P}_m \equiv \mathsf{U}(\mathfrak{P}^{-m_0} | m_0 \geqq m),$$

is the class of all elements each for some stage $m_0 \geqq m$ not developed, the least common superclass of the classes $\mathfrak{P}^{-m_0}(m_0 \geqq m)$.

As to an element p the indices l of existent classes of stage m are designated by the notations:

$$g^p; \qquad h^p,$$

according as the class $\mathfrak{P}^{ml}$ contains or does not contain the element p. Then *the class* $\mathfrak{P}^{pm}$:

$$\mathfrak{P}^{pm} \equiv \mathsf{U}(\mathfrak{P}^{mg^p} | g^p),$$

is the least common superclass of the classes of stage m which contain p, the class of all elements each of which is by stage m of the development *directly connected* with the element p, in that they belong to the same class of stage m. *The class* $\mathfrak{P}_{pm}$:

$$\mathfrak{P}_{pm} \equiv \mathsf{U}(\mathfrak{P}^{m_0}, m_0 \geqq m),$$

is the least common superclass of the classes $\mathfrak{P}^{pm_0}(m_0 \geqq m)$, the class of all elements each of which is by some stage $m_0 \geqq m$ of the development directly connected with the element p.

* I. G. A., § 76.

15. *Some special developments.*—The *ordinary developments** of $\mathfrak{P}^{\mathrm{I}} \equiv 1$; $\mathfrak{P}^{\mathrm{II}_n} \equiv 1, 2, \cdots, n$; $\mathfrak{P}^{\mathrm{III}} \equiv 1, 2, \cdots, n, \cdots$; $\mathfrak{P}^{\mathrm{IV}} \equiv$ interval $(0\ 1)$, respectively denoted by:

$$\Delta^{\mathrm{I}};\ \Delta^{\mathrm{II}_n};\ \Delta^{\mathrm{III}};\ \Delta^{\mathrm{IV}},$$

may be indicated as follows:

$\Delta^{\mathrm{I}} : \equiv : \Delta \ni m \,.\supset.\, l_m = 1 \,.\, \mathfrak{P}^{ml} = 1;$

$\Delta^{\mathrm{II}_n} : \equiv : \Delta \ni m \,.\supset.\, l_m = n \,.\, \mathfrak{P}^{ml} = l \ \ (l);$

$\Delta^{\mathrm{III}} : \equiv : \Delta \ni m \,.\supset.\, l_m = m \,.\, \mathfrak{P}^{ml} = l \ \ (l);$

$\Delta^{\mathrm{IV}} : \equiv : \Delta \ni m \,.\supset.\, l_m = m + 1$. for every l the class $\mathfrak{P}^{ml}$ is the interval $\left(\frac{l-2}{m}\ \frac{l}{m}\right)$ where $-\frac{1}{m}$ is taken as 0 and $\frac{m+1}{m}$ is taken as 1.

16. *Properties of developments* Δ *derived from properties of stages* Δ^m.—Consider a property P of stages Δ^m of a development Δ of a class $\mathfrak{P}$, in particular one of the three properties:

(1) $$P \equiv V\ (\textit{vacuous});\quad F\ (\textit{fine});\quad C\ (\textit{complete}),$$

which are about to be defined.

A stage Δ^m of the development Δ of the class $\mathfrak{P}$ is said to be vacuous; fine; complete, respectively in case $\mathfrak{P}^m$ is null-class; no two elements of $\mathfrak{P}^m$ belong to the same class $\mathfrak{P}^{ml}$; $\mathfrak{P}^m = \mathfrak{P}$.

A property P of stages Δ^m gives rise to certain derived properties of the development Δ. We consider three such derived properties, in notation:

(2) $$P;\quad P_1;\quad P_2.$$

(3) $$\Delta^P : \equiv : \Delta \ni m \,.\supset.\, \Delta^{mP},$$

viz., the development Δ has the property P in case for every m the stage Δ^m has the property P.

(4) $$\Delta^{P_1} .\equiv.\ \Delta \ni (\exists\, m_0 \ni m > m_0 \,.\supset.\, \Delta^{mP}),$$

viz., the development Δ has the property P_1 in case there exists an m_0 such that for every $m > m_0$ the stage Δ^m has the property P. In this case the development is said to possess the property P *ultimately*.

* Cf. I. G. A., §§ 66*a*; 67*a*.

In the theory it is only the ultimate character of the development that is of importance.

(5) $$\Delta^{P_2} . \equiv . \Delta \ni (m . \supset . \exists m_0 > m \ni \Delta^{m_0 P}),$$

viz., the development Δ has the property P_2 in case for every m there exists an $m_0 > m$ such that the stage Δ^{m_0} has the property P.

Thus we have the properties,*

V (vacuous), V_1 (ultimately vacuous), V_2; F (fine), F_1 (ultimately fine), F_2; C (complete), C_1 (ultimately complete), C_2,

of developments Δ, derived from the properties (1) of stages Δ^m. The developments Δ^{I}, Δ^{II_n}, Δ^{III} of §15 are fine developments while the development Δ^{IV} of the same section is a complete development.

In various propositions and proofs of the sequel the negative of such properties P, P_1, P_2 enter. These are in notation: ^-P, $^-P_1$, $^-P_2$ and accordingly:

(6) $$\Delta^{-P} . \sim . \Delta \ni (\exists m \ni \Delta^{m-P}),$$

(7) $$\Delta^{-P_1} . \sim . \Delta \ni (m . \supset . \exists m_0 > m \ni \Delta^{m-P}),$$

(8) $$\Delta^{-P_2} . \sim . \Delta \ni (\exists m_0 \ni m > m_0 . \supset . \Delta^{m-P}).$$

The properties ^-P, $^-P_1$, $^-P_2$ must not be confused with the properties (^-P), $(^-P)_1$, $(^-P)_2$ concerning which, according to the general definitions, we have:

(9) $$\Delta^{(-P)} . \sim . \Delta \ni m . \supset . \Delta^{m-P},$$

(10) $$\Delta^{(-P)_1} . \sim . \Delta \ni (\exists m_0 \ni m > m_0 . \supset . \Delta^{m-P}),$$

(11) $$\Delta^{(-P)_2} . \sim . \Delta \ni (m . \supset . \exists m_0 > m \ni \Delta^{m-P}).$$

16*a*. *Propositions.*—

(1) $\Delta^P . \supset . \Delta^{P_1}$.

(2) $\Delta^{P_1} . \supset . \Delta^{P_2}$.

(3) $\Delta^{P_2} . \sim . \Delta^{(-P)_1}$.

(4) $\Delta^V . \supset . \Delta^{F-C}$.

(5) $\Delta^{V_1} . \supset . \Delta^{F_1-C_1}$.

(6) $\Delta^{V_2} . \supset . \Delta^{F_2}$.

17. *Developmental systems*† $\mathfrak{D}(\mathfrak{M})$ *relative to a class* $\mathfrak{M}$ *of functions and a development* Δ.—A developmental system $\mathfrak{D}(\mathfrak{M})$:

(1) $$\mathfrak{D}(\mathfrak{M}) \equiv ((\delta^{ml})),$$

* The properties fine and complete of developments Δ are defined by Moore in I. G. A. § 67*b*. The property vacuous is likewise defined in I. G. A., § 75.

† I. G. A., § 78.

is a system of functions δ^{ml} on $\mathfrak{P}$ for which there exists a representative system $\mathfrak{R}^{\Delta} \equiv ((r^{ml}))$ of the development Δ such that

$$(1a) \quad e : \supset : \exists\, m_e \ni (m \geqq m_e \,.\, p) \,.\supset.\, A(\sum_g \delta_p^{mg} - 1) \leqq e \,.\, A(\sum_g A\delta_p^{mq} - 1) \leqq e;$$

$$(1b) \quad \mu \,.\supset.\, \exists\, \mu_0 \ni \Big(e : \supset : \exists\, m_{\mu e} \ni (m \geqq \breve{m}_{\mu e} \,.\, p) \,.\supset. \sum_h A\mu_{r^{mh}}\, \delta_p^{mh} \leqq eA\mu_{0p}\Big).$$

Here the indices g, h, are understood to be g^p, h^p; and the conditions (1a), (1b) are understood to apply only in so far as such indices are available, that is, for the various stages m the condition (1a) applies only for those elements p which belong to at least one class of stage m, and the condition (1b) applies only for those elements p which belong not to every existent class of stage m. The condition (1a) is an absolute condition, and (1b) is relative to the class $\mathfrak{M}$.

The system $\mathfrak{D}(\mathfrak{M})$ need not actually depend upon the class $\mathfrak{M}$. A system $\mathfrak{D}$:

$$(1') \qquad \mathfrak{D} \equiv ((\delta^{ml})),$$

satisfying (1a) and

$$(1'b) \qquad m \,.\, p \,.\, h^p \,.\supset.\, \delta_p^{mh^p} = 0,$$

is a system $\mathfrak{D}(\mathfrak{M})$ as to every class $\mathfrak{M}$. Such are most of the developmental systems given later in this paper.*

We notice that in case the development Δ is vacuous at stage m then both (1a) and (1b) are holding vacuously for that stage, i. e., for no p is there an available index g^p and there are no representative elements r^{mh}.

18.† *The property* Δ *of classes* $\mathfrak{M}$ *of systems* $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$.‡—A class $\mathfrak{M}$ of a system $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$ is said to have the property Δ in case there exists a developmental system $\mathfrak{D}(\mathfrak{M})$ belonging to $\mathfrak{M}$. In symbols

$$\mathfrak{M}^{\Delta} \,.\equiv.\, \mathfrak{M} \ni (\exists\, \mathfrak{D}(\mathfrak{M}) \ni \mathfrak{D}(\mathfrak{M})^{B_0\mathfrak{M}}).$$

18a. *Propositions.*—

$$(1) \qquad \mathfrak{M}^{\Delta} \,.\supset.\, (\mathfrak{A}\mathfrak{M})^{\Delta}.$$

$$(2) \qquad \mathfrak{M}^{D_1\Delta} \,.\supset.\, \mathfrak{M}_L{}^{D_1\Delta}.$$

$$(3)\S \qquad \mathfrak{M}^{D_1\Delta} \,.\supset.\, \mathfrak{M}_*{}^{D_1\Delta}.$$

* § 48.

† I. G. A., § 79.

‡ We speak of a class $\mathfrak{P}$, a development Δ of $\mathfrak{P}$, a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$, as a system $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$.

§ In I. G. A., § 79 (3) is given with D instead of D_1, but D_1 is all that is necessary.

19. *Relations K between the elements of a class $\mathfrak{P}$ with respect to a development Δ.*—We define at once what we mean by the symbols* K_{pm}, $K_{p_1p_2m}$.

$$K_{pm} . \equiv . (pm) \ni \exists m_0 \geqq m \ni p^{-m_0};$$

$$K_{p_1p_2m} . \equiv . (p_1p_2m) \ni \exists (m_0 \geqq m . l_0 \leqq l_{m_0}) \ni (p_1^{m_0 l_0} . p_2^{m_0 l_0}).$$

that is,

K_{pm} denotes an element p for some stage $m_0 \geqq m$ not developed; $K_{p_1p_2m}$ denotes two elements p_1p_2 (not necessarily distinct) belonging to the same class of some stage $m_0 \geqq m$.

Relative to classes $\mathfrak{P}'$; $\mathfrak{P}''$; etc., we denote relations above as

$$K'_{p'm}, \ K'_{p_1'p_2'm}; \ K''_{p''m}, \ K''_{p_1''p_2''m}; \text{ etc.}$$

In the ordinary development† of $\mathfrak{P}^{\text{III}}$ the relation K_{pm} is equivalent to the relation $p > m$ and in the ordinary development of $\mathfrak{P}^{\text{IV}}$ the relation $K_{p_1p_2m}$ is equivalent to the relation $A(p_1 - p_2) < 1/m$.

20. *Properties*‡ K_1, K_2, K_{12} *of classes $\mathfrak{M}$ of functions of systems* $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$.—We define directly the properties $K_1\mathfrak{M}$; $K_2\mathfrak{M}$; $K_{12}\mathfrak{M}$ of φ, where φ is a function and $\mathfrak{M}$ a class of functions on $\mathfrak{P}$ to $\mathfrak{A}$. The definitions are relative to a development Δ of $\mathfrak{P}$.

$$(1) \qquad \varphi^{K_1\mathfrak{M}} . \equiv . \varphi \ni (\exists \mu \ni e : \supset : \exists m_e \ni K_{pm_e} . \supset . A\varphi_p \leqq eA\mu_p),$$

viz., the function φ has the property $K_1\mathfrak{M}$ in case φ is such that there exists a function μ of $\mathfrak{M}$ such that for every positive number e there exists a stage m_e of the development such that the relation

$$K_{pm_e}$$

implies

$$A\varphi_p \leqq eA\mu_p.$$

$$(2) \quad \varphi^{K_2\mathfrak{M}} . \equiv . \varphi \ni (\exists \mu \ni e : \supset : \exists m_e \ni K_{p_1p_2m_e} . \supset . A(\varphi_{p_1} - \varphi_{p_2}) \leqq eA\mu_{p_1}),$$

viz., the function φ has the property $K_2\mathfrak{M}$ in case φ is such that there exists a function μ of $\mathfrak{M}$ such that for every positive number e there exists a stage m_e of the development such that the relation

$$K_{p_1p_2m_e}$$

* The relations here defined are the relations K_1^Δ, K_2^Δ of I. G. A., § 77. In § 69 and following of the work mentioned Moore discusses general K relations of which the relations K^Δ are special instances.

† Cf. § 15.

‡ The properties here defined are the properties K_1^Δ, K_2^Δ, K_{12}^Δ of I. G. A., § 77, and are special cases of the more general K properties discussed by Moore.

implies

$$A(\varphi_{p_1} - \varphi_{p_2}) \leqq eA\mu_{p_1}.$$

(3) $$\varphi^{K_{12}\mathfrak{M}} . \equiv . \varphi^{K_1\mathfrak{M}K_2\mathfrak{M}}.$$

A class $\mathfrak{F} \equiv [\varphi]$ of functions is said to have the property $K_i\mathfrak{M}$ $(i = 1, 2, 12)$ in case each function φ of $\mathfrak{F}$ has the property $K_i\mathfrak{M}$. In symbols:

(4_i) $$\mathfrak{F}^{K_i\mathfrak{M}} : \equiv : \mathfrak{F} \ni \varphi . \supset . \varphi^{K_i\mathfrak{M}} \qquad (i = 1, 2, 12).$$

In case of a function φ' and a class $\mathfrak{M}'$ of functions on $\mathfrak{P}'$ to $\mathfrak{A}$ we use the notations:

$$\varphi'^{K_1'\mathfrak{M}'}; \quad \varphi'^{K_2'\mathfrak{M}'}; \quad \varphi'^{K_{12}'\mathfrak{M}'}.$$

We give also the definitions of the properties $K_1'\mathfrak{M}'(\mathfrak{M}'')$; $K_2'\mathfrak{M}'(\mathfrak{M}'')$; $K_{12}'\mathfrak{M}'(\mathfrak{M}'')$ of a function φ on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$, where $\mathfrak{M}'$, $\mathfrak{M}''$ are classes of functions on $\mathfrak{P}'$, $\mathfrak{P}''$, respectively, to $\mathfrak{A}$. The definitions are relative to a development Δ' of $\mathfrak{P}'$.

(5) $$\varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')} . \equiv . \varphi \ni \big(\exists (\mu', \mu'') \ni e : \supset : \exists m_e \ni K'_{p'm_e} . \supset . A\varphi'_{p'} \leqq eA\mu'_{p'}\mu'' \big),$$

viz., the function φ has the property $K_1'\mathfrak{M}'(\mathfrak{M}'')$ in case φ is such that there exists a function μ' of $\mathfrak{M}'$ and a function μ'' of $\mathfrak{M}''$ such that for every e there exists an m_e such that the relation

$$K'_{p'm_e}$$

implies

$$A\varphi'_{p'} \leqq eA\mu'_{p'}\mu''$$

i. e.,

$$A\varphi'_{p'p''} \leqq eA\mu'_{p'}\mu''_{p''} \quad (p'').$$

(6) $$\varphi^{K_2'\mathfrak{M}'(\mathfrak{M}'')} . \equiv . \varphi \ni \big(\exists (\mu', \mu'') \ni e : \supset : \exists m_e \ni K'_{p_1'p_2'm_e} . \supset . A(\varphi_{p_1'} - \varphi_{p_2'}) \leqq eA\mu'_{p_1'}\mu'' \big).$$

(7) $$\varphi^{K_{12}'\mathfrak{M}'(\mathfrak{M}'')} . \equiv . \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')K_2'\mathfrak{M}'(\mathfrak{M}'')}.$$

A class $\mathfrak{F} \equiv [\varphi]$ of functions on $\mathfrak{P}'\mathfrak{P}''$ is said to have the property $K_i'\mathfrak{M}'(\mathfrak{M}'')$, $(i = 1, 2, 12)$, in case each function φ of the class $\mathfrak{F}$ has the property $K_i'\mathfrak{M}'(\mathfrak{M}'')$, in symbols:

(8) $$\mathfrak{F}^{K_i'\mathfrak{M}'(\mathfrak{M}'')} \equiv \mathfrak{F} \ni \varphi . \supset . \varphi^{K_i'\mathfrak{M}'(\mathfrak{M}'')} \qquad i = (1, 2, 12).$$

A class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$ is said to have the property

$$K_1; \; K_2; \; K_{12},$$

respectively in case each function μ of $\mathfrak{M}$ has the property

$$K_1\mathfrak{M};\ K_2\mathfrak{M};\ K_{12}\mathfrak{M}.$$

In symbols:

$$(9) \qquad \mathfrak{M}^{K_1} :\equiv : \mathfrak{M}^{K_1\mathfrak{M}} :\equiv : \mu \,.\supset.\, \mu^{K_1\mathfrak{M}};$$

$$(10) \qquad \mathfrak{M}^{K_2} :\equiv : \mathfrak{M}^{K_2\mathfrak{M}} :\equiv : \mu \,.\supset.\, \mu^{K_2\mathfrak{M}};$$

$$(11) \qquad \mathfrak{M}^{K_{12}} :\equiv : \mathfrak{M}^{K_{12}\mathfrak{M}} :\equiv : \mu \,.\supset.\, \mu^{K_{12}\mathfrak{M}}.$$

The properties K_1; K_2; K_{12} are properties of special reference in that they refer to a development Δ of the class $\mathfrak{P}$.

Relative to the ordinary developments* of $\mathfrak{P}^{\mathrm{I}}$; $\mathfrak{P}^{\mathrm{II}_n}$; $\mathfrak{P}^{\mathrm{III}}$; $\mathfrak{P}^{\mathrm{IV}}$, the classes $\mathfrak{M}^{\mathrm{I}}$; $\mathfrak{M}^{\mathrm{II}_n}$; $\mathfrak{M}^{\mathrm{III}}$; $\mathfrak{M}^{\mathrm{III}_0}$; $\mathfrak{M}^{\mathrm{III}_s}$; $\mathfrak{M}^{\mathrm{IV}}$ possess the properties K_1, K_2, K_{12}.

20*a*. *Propositions.*—

(1)† $$\Delta^{G_1} \,.\, \mathfrak{M} \,.\supset.\, \mathfrak{M}^{K_1},$$

(2) $$\Delta^{F_1} \,.\, \mathfrak{M} \,.\supset.\, \mathfrak{M}^{K_2},$$

(3) $$\Delta^{V_1} \,.\, \mathfrak{M} \,.\supset.\, \mathfrak{M}^{K_2}.$$

In (1, 2, 3) the properties K_1; K_2 are said to hold vacuously, viz., in (1) there exists a stage m_0 such that for m exceeding m_0 there is no element p satisfying the relation K_{pm}; and in (2) there exists an m_0 such that for m exceeding m_0 no two elements p_1p_2 satisfy the relation $K_{p_1p_2m}$. The property V_1 is, by § 16*a*6, a special case of F_1.

Similarly if φ is a function on $\mathfrak{P}'\mathfrak{P}''$ and $\mathfrak{M}'$ a class of functions on $\mathfrak{P}'$ and $\mathfrak{M}''$ a class of functions on $\mathfrak{P}''$ we have the propositions:

$$(1') \qquad \Delta'^{G_1} \,.\, \varphi \,.\, \mathfrak{M}' \,.\, \mathfrak{M}'' \,.\supset.\, \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')};$$

$$(2') \qquad \Delta'^{F_1} \,.\, \varphi \,.\, \mathfrak{M}' \,.\, \mathfrak{M}'' \,.\supset.\, \varphi^{K_2'\mathfrak{M}'(\mathfrak{M}'')};$$

$$(3') \qquad \Delta'^{V_1} \,.\, \varphi \,.\, \mathfrak{M}' \,.\, \mathfrak{M}'' \,.\supset.\, \varphi^{K_2'\mathfrak{M}'(\mathfrak{M}'')}.$$

21. *Composition of developments*‡ Δ.—Consider two developments:

$$\Delta' \equiv ((\mathfrak{P}'^{m_l'})); \qquad \Delta'' \equiv ((\mathfrak{P}''^{m_l''})),$$

with corresponding sequences:

$$\{l_m'\}; \qquad \{l_m''\},$$

* § 15.

† Every class $\mathfrak{M}$ of functions has the property K_1; K_2; K_2, with respect to an ultimately complete; ultimately fine; ultimately vacuous development Δ.

‡ I. G. A., § 83.

of the respective classes:

$$\mathfrak{P}'; \qquad \mathfrak{P}''.$$

By the composition* for every stage $\mathfrak{M}$ of the l'_m classes of stage m of Δ' with the l''_m classes of stage m of Δ'' we secure the

$$l_m \equiv l'_m l''_m$$

classes of stage m of a development Δ, in notation:

$$\Delta'\Delta'' \quad \text{or} \quad \Delta'',$$

of the class:

$$\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}'',$$

the *composite development* $\Delta'\Delta'' \equiv \Delta''$ of the developments Δ', Δ''. For notation it is convenient to use for index l^m or l of stage m the double index $l'^m l''^m$ or $l'l''$. Then

$$\Delta'\Delta'' \equiv \Delta'' \equiv ((\mathfrak{P}^{ml})) \equiv ((\mathfrak{P}^{ml'l''}))$$

where

$$l_m \equiv l'_m l''_m \ (m); \quad \mathfrak{P}^{ml'l''} \equiv \mathfrak{P}'^{ml'}\ \mathfrak{P}''^{ml''} \ (ml'l'').$$

Similarly, from two representative systems:

$$\mathfrak{R}' \equiv ((r'^{ml'})); \quad \mathfrak{R}'' \equiv ((r''^{ml''})),$$

of Δ', Δ'' respectively we obtain a *composite system*:

$$\mathfrak{R}'\mathfrak{R}'' \equiv \mathfrak{R}'' \equiv ((r^{ml})) \equiv ((r^{ml'l''})),$$

where

$$r^{ml''l'} \equiv r'^{ml'} r''^{ml''} \ (ml'l'),$$

in fact, a representative system of the composite development $\Delta'\Delta'' \equiv \Delta''$, the *composite representative system* $\mathfrak{R}'\mathfrak{R}'' \equiv \mathfrak{R}''$ of the representative systems $\mathfrak{R}'$, $\mathfrak{R}''$.

Similarly from two developmental systems:

$$\mathfrak{D}' \equiv \mathfrak{D}'(\mathfrak{M}') \equiv ((\delta'^{ml'})); \qquad \mathfrak{D}'' \equiv \mathfrak{D}''(\mathfrak{M}'') \equiv ((\delta''^{ml''})),$$

relative to $(\Delta', \mathfrak{M}')$, $(\Delta'', \mathfrak{M}'')$ respectively we obtain by multiplication a *composite system*:

$$\mathfrak{D}'\mathfrak{D}'' \equiv \mathfrak{D}'' \equiv ((\delta^{ml})) \equiv ((\delta^{ml'l''})),$$

where

$$\delta^{ml'l''} \equiv \delta'^{ml'}\delta''^{ml''} \ (ml'l').$$

In case $\mathfrak{M}'$ has the properties $D_1\Delta' K'_2$, and $\mathfrak{M}''$ the properties $D_1\Delta'' K''_2$

* The composite of two classes is non-existent if either component is non-existent.

the composite developmental system $\mathfrak{T}'''$ is provably a developmental system as to $(\Delta'\Delta'', \mathfrak{M}'\mathfrak{M}'')$.

In the composite development we see that

$$(p'p'')^m . \infty . p'^m . p''^m; \quad (p'p'')^{-m} . \infty . p'^{-m} \smile p''^{-m}.$$

21*a*. *Propositions.*—Using the previous notation and the notations K_i'; K_i'', K_i $(i = 1, 2, 12)$ for properties of classes of functions on $\mathfrak{P}'$; $\mathfrak{P}''$; $\mathfrak{P}$ respectively, we have the propositions:

(1)* $\quad \mathfrak{M}'^{D_1\Delta'K_2'} . \mathfrak{M}''^{D_1\Delta''K_2''} . \supset . (\mathfrak{M}'\mathfrak{M}'')^{\Delta''}.$

(2)† $\quad \mathfrak{M}'^{D_1\Delta'K_2'} . \mathfrak{M}''^{D_1\Delta''K_2''} . \supset . (\mathfrak{M}'\mathfrak{M}'')_L^{D_1\Delta''} . (\mathfrak{M}'\mathfrak{M}'')_*^{D_1\Delta''}.$

(3)‡ $\quad \mathfrak{M}'^{D_1K_1'} . \mathfrak{M}''^{D_1K_1''} . \supset . (\mathfrak{M}'\mathfrak{M}'')^{D_1K_1}.$

(4) $\quad \mathfrak{M}'^{D_1K_2'} . \mathfrak{M}''^{D_1K_2''} . \supset . (\mathfrak{M}'\mathfrak{M}'')^{D_1K_2}.$

(5) $\quad \Delta'^{V_i} . \Delta'' . \supset . \Delta'''^{V_i} \qquad (V_i = V, V_1, V_2).$

(6) $\quad \Delta'^{F} . \Delta''^{F} . \supset . \Delta'''^{F}.$

(7) $\quad \Delta'^{F_1} . \Delta''^{F_1} . \supset . \Delta'''^{F_1}.$

(8) $\quad \Delta'^{C} . \Delta''^{C} . \supset . \Delta'''^{C}.$

(9) $\quad \Delta'^{C_1} . \Delta''^{C_1} . \supset . \Delta'''^{C_1}.$

22. *Reduction of developments* Δ.—The reduction of a class of elements $\mathfrak{P}$ with a development Δ leads to a reduced development Δ_R of $\mathfrak{P}_R$. We use the index m_R and the bipartite index m_Rl_R relative to the development Δ_R. These indices have the same range of values as the corresponding indices m and ml of the development Δ. Then the development Δ_R of $\mathfrak{P}_R$ is completely defined by:

$$\mathfrak{P}_R^{ml} \equiv \mathfrak{P}_R^{m_Rl_R} \equiv \cap(\mathfrak{P}_R, \mathfrak{P}^{ml}) \quad (mlm_Rl_R).$$

Thus

$$\mathfrak{P}_R^{m} \equiv \mathfrak{P}_R^{m_R} = \cap(\mathfrak{P}_R, \mathfrak{P}^{m}) \quad (m_R).$$

In case a reduced development Δ_R possesses a property P, then the development Δ is said to possess the property P on $\mathfrak{P}_R$ in notation:§

* I. G. A., § 83, p. 147, as revised.

† Cf. § 19.3.

‡ In I. G. A., § 72*b*3$_1'$, 3$_2'$ similar propositions are given in a more general setting.

§ Cf. § 11*a* for similar notations as to properties of functions and classes of functions

$\Delta^{P(\mathfrak{P}_R)}$. Thus from § 16 we have the properties $V(\mathfrak{P}_R)$, $V_1(\mathfrak{P}_R)$, $V_2(\mathfrak{P}_R)$, $F(\mathfrak{P}_R)$, $F_1(\mathfrak{P}_R)$, $F_2(\mathfrak{P}_R)$, $C(\mathfrak{P}_R)$, $C_1(\mathfrak{P}_R)$, $C_2(\mathfrak{P}_R)$ of developments Δ where $\mathfrak{P}_R$ is any reduction of $\mathfrak{P}$. It should be noted that any class $\mathfrak{P}_1$ of $\mathfrak{P}$ is a reduction of $\mathfrak{P}$.

We note the following proposition relative to a class $\mathfrak{P}$ with a development Δ and a class $\mathfrak{P}_R$ with a development Δ_R:

$$(1) \qquad \theta^{K_i\mathfrak{M}} . \supset . \theta_R^{K_i\mathfrak{M}_R}, \qquad (i = 1, 2, 12).$$

23. *The properties* K_{1*}, K_{2*}, K_{12*} of classes $\mathfrak{M}$ of functions of systems* $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$.—In the following definitions $\mathfrak{M}'$, $\mathfrak{M}''$ are classes of functions on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, φ is a function on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$, and $(\mathfrak{M}'\mathfrak{M}'')_*$ is the $*$-extension of the composite class $(\mathfrak{M}'\mathfrak{M}'')$.

$$(1) \quad \mathfrak{M}'^{K_1'*} :\equiv : \mathfrak{M}' \ni \mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{M}''(p')}];$$

$$(2) \quad \mathfrak{M}'^{K_2'*} :\equiv : \mathfrak{M}' \ni \mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \varphi^{K_2'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_1\mathfrak{M}''(p')}];$$

$$(3) \quad \mathfrak{M}'^{K_{12}'*} . \equiv . \mathfrak{M}' \ni \mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \varphi^{K_{12}'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{M}''(p')}];$$

viz., $\mathfrak{M}'$ has the property K_{i*}', $(i = 1, 2, 12)$, in case $\mathfrak{M}'$ is such that for every $\mathfrak{M}''^{LCD}$ the $*$-extension of the class $\mathfrak{M}'\mathfrak{M}''$ is the same as the class of all functions φ having the property $K_i'\mathfrak{M}'(\mathfrak{M}'')$ and belonging to $\mathfrak{M}''$ for every p'.

23*a*. *Propositions† concerning relations and classes of functions on* $\mathfrak{P}$. The following propositions relative to the class $\mathfrak{P}$ with a general development Δ are for $i = 1, 2, 12$.

$$(1_i) \qquad \mathfrak{R}^{K_i\mathfrak{M}} . \supset . (A\mathfrak{R})^{K_i\mathfrak{M}} . (\mathfrak{A}\mathfrak{R})^{K_i\mathfrak{M}}.$$

$$(2_i) \qquad \mathfrak{M}_1^{B_0\mathfrak{M}_2} . \mathfrak{R}^{K_i\mathfrak{M}_1} . \supset . \mathfrak{R}^{K_i\mathfrak{M}_2}.$$

$$(3_i) \qquad \mathfrak{M}_1^{B_1\mathfrak{A}\mathfrak{M}_2} . \mathfrak{R}^{K_i\mathfrak{M}_1} . \supset . \mathfrak{R}^{K_i\mathfrak{M}_2}.$$

$$(3_1') \qquad \mathfrak{R}^{K_1\mathfrak{M}} . \mathfrak{Z}^{B_1\mathfrak{A}\mathfrak{R}} . \supset . \mathfrak{Z}^{K_1\mathfrak{M}}.$$

$$(4_i) \qquad \mathfrak{M}^{D_1} . \mathfrak{R}^{K_i\mathfrak{M}} . \supset . \mathfrak{R}_L{}^{K_i\mathfrak{M}}.$$

$$(5_i) \qquad \mathfrak{M}^{D} . \mathfrak{R}^{K_i\mathfrak{M}} . \mathfrak{Z}^{K_i\mathfrak{M} \,.\, B_1\mathfrak{A}\mathfrak{M}} . \supset . \mathfrak{R}_{\mathfrak{Z}}{}^{K_i\mathfrak{M}}.$$

$$(6_i) \qquad \mathfrak{M}^{D} . \mathfrak{R}^{K_i\mathfrak{M} \,.\, B_1\mathfrak{A}\mathfrak{M}} . \supset . \mathfrak{R}_L{}^{K_i\mathfrak{M} \,.\, B_1\mathfrak{A}\mathfrak{M}} . \mathfrak{R}_*{}^{K_i\mathfrak{M} \,.\, B_1\mathfrak{A}\mathfrak{M}}.$$

$$(7_i) \qquad \mathfrak{M}^{DK_i} . \supset . \mathfrak{M}_L{}^{DK_i} . \mathfrak{M}_{\mathfrak{M}}{}^{DK_i} . \mathfrak{M}_*{}^{DK_i}.$$

* Cf. I. G. A., § 72.

† Propositions 1_i–14_i are given in I. G. A., § 72*a* as relative to K relations and properties more general than those derived from the development Δ.

$$(8_i)\qquad \mathfrak{M} \cdot \mathfrak{F}_i \equiv [\text{all } \varphi^{K_i\mathfrak{M} \cdot B_1\mathfrak{A}\mathfrak{M}}] : \supset : \mathfrak{F}_i{}^{K_i\mathfrak{M} \cdot A} : \mathfrak{F}_i = \mathfrak{A}\mathfrak{F}_i : \mathfrak{M}^{D_1} \cdot \supset \cdot \mathfrak{F}_i{}^{L} : \mathfrak{M}^{D} \cdot \supset \cdot \mathfrak{F}_i{}^{LC_*} : \mathfrak{M}^{K_i} \cdot \supset \cdot \mathfrak{M}^{B_0\mathfrak{F}_i} \cdot \mathfrak{F}_i{}^{K_i}.$$

$$(9_i)\qquad \mathfrak{M}^{D} \cdot \supset \cdot (\mathfrak{M}^{K_i} \cdot \sim \cdot \mathfrak{M}_*{}^{K_i}) \cdot (K_i\mathfrak{M} \cdot \sim \cdot K_i\mathfrak{M}_*).$$

$$(10_i)\qquad \mathfrak{F}^{K_i\mathfrak{M}(\mathfrak{A})} \cdot \sim \cdot \mathfrak{F}^{K_i\mathfrak{M}}.$$

$$(11_i)\qquad \mathfrak{M}^{K_{i*}} \cdot \supset \cdot \mathfrak{M}_* = [\text{all } \varphi^{K_i\mathfrak{M}}] \cdot \mathfrak{M}_*{}^{K_i\mathfrak{M} \cdot A} \cdot \mathfrak{M}^{K_i} \cdot (B_0\mathfrak{M}_* \sim K_i\mathfrak{M}).$$

$$(12_i)\qquad \mathfrak{M}^{DK_{i*}} \cdot \supset \cdot \mathfrak{M}^{K_i} \cdot \mathfrak{M}_L{}^{LDK_iK_{i*} \cdot B_1\mathfrak{A}\mathfrak{M}} \cdot \mathfrak{M}_*{}^{LCDAK_iK_{i*} \cdot B_1\mathfrak{A}\mathfrak{M}}.$$

$$(13_i)\qquad \mathfrak{M}^{D} : \supset : \mathfrak{M}^{K_{i*}} \cdot \sim \cdot \mathfrak{M}_L{}^{K_{i*}} \cdot \sim \cdot \mathfrak{M}_*{}^{K_{i*}}.$$

$$(14_i)\qquad \mathfrak{M}^{LCK_{i*}} \cdot \supset \cdot \mathfrak{M}^{K_iA} \cdot (B_0\mathfrak{M} \cdot \sim \cdot K_i\mathfrak{M}).$$

$$(15)^*\qquad \mathfrak{M}^{D_1\Delta} : \supset : \varphi^{K_{12}\mathfrak{M}} \cdot \supset \cdot \varphi^{B_0\mathfrak{M}_*}.$$

$$(16)\dagger\qquad \mathfrak{M}'^{D_1\Delta'} \cdot \mathfrak{M}''^{D_1} : \supset : \varphi^{K_{12}'\mathfrak{M}'(\mathfrak{M}'') \cdot B_0\mathfrak{A}\mathfrak{M}''(p')} \cdot \supset \cdot \varphi^{B_1\mathfrak{A}\mathfrak{M}' \cdot \mathfrak{M}'' \cdot B_{1*}\mathfrak{M}'\mathfrak{M}''_*}.$$

$$(17)\ddagger\qquad \mathfrak{M}^{D\Delta K_1K_2} : \supset : \mathfrak{M}^{K_{12*}} \cdot \mathfrak{M}_* = [\text{all } \varphi^{K_{12}\mathfrak{M}}] \cdot \mathfrak{M}_*{}^{LCDA\Delta K_{12}K_{12*}B_1\mathfrak{A}\mathfrak{M}}.$$

$$(18)\S\qquad \mathfrak{M}^{LCD\Delta K_1K_2} \cdot \supset \cdot \mathfrak{M}^{AK_{12*}} \cdot \mathfrak{M} = [\text{all } \varphi^{K_{12}\mathfrak{M}}].$$

24. *Resumé.*—We have indicated the fundamental notions of the General Analysis of Moore in so far as they are essential for the present investigation. In particular we have defined the properties:

$$L;\ C;\ D;\ A;\ \Delta;\ K_1;\ K_2;\ K_{12*};\ K_{1*};\ K_{2*};\ D_1,$$

of classes $\mathfrak{M}$ of functions of systems $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$. Of these

$$L;\ C;\ D;\ A;\ D_1$$

are of general reference|| and

$$\Delta;\ K_1;\ K_2;\ K_{12*};\ K_{1*};\ K_{2*}$$

are of special reference,‡ being defined relative to a development Δ of the class $\mathfrak{P}$.

In § 16 we have defined certain special types of developments Δ of which we will make frequent use.

* I. G. A., § 80I.
† I. G. A., § 80II.
‡ I. G. A., § 81II.
§ I. G. A., § 81III.
|| See preface.

Propositions and theorems relative to systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$ *where* $\mathfrak{P}$ *is unconditioned except by the development* Δ. *Theorems relative to the closure of certain genera of systems* $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$.

§§ 25–40.

25. The following propositions are relative to a general class $\mathfrak{P}$ and to classes $\mathfrak{M}$ and $\mathfrak{N}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$:

(1)* $$\mathfrak{M}^{B_1 \mathfrak{A}\mathfrak{N}} . \mathfrak{N}^{B_1 \mathfrak{A}\mathfrak{M}} :\supset : \mathfrak{M}^{D_1} . \sim . \mathfrak{N}^{D_1},$$

(2)* $$\mathfrak{M}^{B_1 \mathfrak{A}\mathfrak{N}} . \mathfrak{N}^{B_1 \mathfrak{A}\mathfrak{M}} :\supset : \mathfrak{M}^{D} . \sim . \mathfrak{N}^{D},$$

viz., if $\mathfrak{M}$ is dominated by $\mathfrak{A}\mathfrak{N}$ and $\mathfrak{N}$ by $\mathfrak{A}\mathfrak{M}$ then the statement that $\mathfrak{M}$ has the property D_1; D, is equivalent to the statement that $\mathfrak{N}$ has the property D_1; D, respectively.

In addition we have the following proposition relative to any development Δ of the class $\mathfrak{P}$:

(3) $$\mathfrak{M}^{B_1 \mathfrak{A}\mathfrak{N}} . \mathfrak{N}^{B_1 \mathfrak{A}\mathfrak{M}} :\supset : \mathfrak{M}^{K_1} . \sim . \mathfrak{N}^{K_1}.$$

26. We have the following simple propositions relative to a class $\mathfrak{P}$ with a development Δ:

(1) $$\theta^{\text{constant } (\mathfrak{P})} . \sigma . \supset . \theta^{K_2 \sigma};$$

(2) $$\mathfrak{M}^{K_2 *} . \supset . [\text{all } \theta^{\text{constant } (\mathfrak{P})}]^{B_0 \mathfrak{M}_*};$$

(3) $$\mathfrak{M} = [0]\dagger . \supset . \mathfrak{M}^{-K_2 *}.$$

In case the development of $\mathfrak{P}'$ is such that there exists a pair of elements $p_1' p_2'$ such that for every stage m there exists a stage $m_0 \geqq m$ such that the elements p_1', p_2' are directly connected at stage m_0, i. e.,

$$\Delta' \ni \left(\exists (p_1',\ p_2') \ni m . \supset . \exists (m_0 > m . l_0 \leqq l_{m_0}) \ni (p_1'^{m_0 l_0} . p_2'^{m_0 l_0})\right)$$

then

(4) $$\sigma' :\supset : \mu'^{K_2' \sigma'} . \supset . \mu'_{p_1'} = \mu'_{p_2'},$$

(5) $$\sigma' \sigma'' :\supset : \varphi^{K_2' \sigma' (\sigma'')} . \supset . \varphi_{p_1' p''} = \varphi_{p_2' p''} \quad (p'').$$

In case the elements p_1', p_2' are each ultimately developed, i. e.,

$$\exists m_0 \ni m > m_0 . \supset . p_1'^m . p_2'^m$$

we have the proposition:

(6) $$\mathfrak{M}' \ni \mu' . \supset . \mu'_{p_1'} = \mu'_{p_2'} :\supset : \mathfrak{M}'^{-K_1' *}.$$

* Cf. I. G. A., § 22*a*2, 2′.

† [0] ≡ class consisting of a single function whose every functional value is 0.

(2) follows from §23a11$_i$ and (1). (3) follows from (2) and the consideration of the class $(\mathfrak{M}'\mathfrak{M}'')_*$ in case $\mathfrak{M}' = [0]$. (6) follows from §23a11$_i$ and the following proposition:

(7) $$\mathfrak{M}' \ni \mu' \,.\supset.\, \mu'_{p_1'} = \mu'_{p_2'} :\supset: \theta'^{B_{p_1'}\mathfrak{M}'}{}_* \,.\supset.\, \theta'_{p_1'} = \theta'_{p_2'}.$$

This last proposition gives rise to a more general one:

(8) $$\mathfrak{M}' \ni (\mu' \,.\supset.\, \mu'_{p_1'} = \mu'_{p_2'}) \,.\, \mathfrak{M}'' :\supset: \theta^{B_{p'}\mathfrak{M}'\mathfrak{M}''}{}_* \,.\supset.\, \theta_{p_1'p''} = \theta_{p_2'p''} \quad (p'').$$

27. *Propositions.*—

(1) $$\Delta^{F_1} :\supset: \mathfrak{M}^{K_1} \,.\infty.\, \mathfrak{M}^{K_{12}}.$$

(2) $$\Delta^{C_1} :\supset: \mathfrak{M}^{K_2} \,.\infty.\, \mathfrak{M}^{K_{12}}.$$

(3) $$\Delta^{V_1} :\supset:. \mathfrak{M}^{K_1} \,.\infty.\, \mathfrak{M}^{K_{12}}.$$

(4) $$\Delta^{V_2} \,.\, \mathfrak{M}^{K_1} \,.\supset.\, \mathfrak{M} = [0].$$

(5) $$\Delta^{V_2} \,.\, \mathfrak{M} :\supset: \mathfrak{M}^{K_{1*}} \,.\infty.\, \mathfrak{M}^{K_{12*}} \,.\infty.\, \mathfrak{M} = [0].$$

(6) $$\Delta'^{F_1} :\supset: \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')} \,.\infty.\, \varphi^{K_{12}'\mathfrak{M}'\mathfrak{M}''}.$$

(7) $$\Delta'^{C_1} :\supset: \varphi^{K_2'\mathfrak{M}'(\mathfrak{M}'')} \,.\infty.\, \varphi^{K_{12}'\mathfrak{M}'\mathfrak{M}''}.$$

(8) $$\Delta'^{V_1} :\supset: \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')} \,.\infty.\, \varphi^{K_{12}'\mathfrak{M}'\mathfrak{M}''}.$$

(9) $$\Delta'^{V_2} :\supset: \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')} \,.\supset.\, \varphi_{p'p''} = 0 \quad (p'p'').$$

(10) $$\Delta^{F_1} :\supset: \mathfrak{M}^{K_{1*}} \,.\infty.\, \mathfrak{M}^{K_{12*}}.$$

(11) $$\Delta^{C_1} :\supset: \mathfrak{M}^{K_{2*}} \,.\infty.\, \mathfrak{M}^{K_{12*}}_{a_*}.$$

(12) $$\Delta^{V_1} :\supset: \mathfrak{M}^{K_{1*}} \,.\infty.\, \mathfrak{M}^{K_{12*}}_{i}.$$

(13) $$\Delta \,.\, \mathfrak{M}^{\Delta} = [0] \,.\supset.\, \Delta^{V_1}.$$

In (1 . . . 5), (10 . . . 13) $\mathfrak{M}$ is a class of functions on $\mathfrak{P}$ with a development Δ conditioned as indicated. In (6 . . . 9) φ is a function on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$, $\mathfrak{M}'$ and $\mathfrak{M}''$ are classes of functions on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, and Δ' is a development of $\mathfrak{P}'$ conditioned as indicated. (1, 2, 3) follow from propositions (1, 2, 3) of §20a, and (6, 7, 8) follow in a like manner from (1′, 2′, 3′) of the same section. (5) follows from (4) and from §23a11$_i$. (10, 11, 12) follow from (6, 7, 8) and the definitions of the properties K'_{1*} K'_{2*} K'_{12*}. Thus the proof of (10) is as follows:

Consider classes:

$$\mathfrak{F}_1 \equiv [\text{all } \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}];$$

$$\mathfrak{F}_{12} \equiv [\text{all } \varphi^{K_{12}'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}],$$

which occur in the definitions of K'_{1*}; K'_{12*} respectively. By (6)

$$\Delta'^{F_1} :\supset : \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')} \,.\, \infty \,.\, \varphi^{K_{12}'\mathfrak{M}'(\mathfrak{M}'')}.$$

$$\therefore \quad \mathfrak{F}_1 = \mathfrak{F}_{12} \quad \text{and} \quad \mathfrak{M}'^{K_1'*} \,.\, \infty \,.\, \mathfrak{M}'^{K_{12}'*}.$$

28. *A relation between the properties* K_{1*}, K_{2*}, K_{12*}.—If a class $\mathfrak{M}'$ of a system $(\mathfrak{A}; \mathfrak{P}'; \Delta'; \mathfrak{M}')$ has the properties K'_{1*} K'_{2*} then the classes* $\mathfrak{F}_1$ and $\mathfrak{F}_2$ defined relative to an arbitrary class $\mathfrak{M}''^{LCD}$, occurring in the definitions† of K'_{1*} K'_{2*}, are each the same as the class $(\mathfrak{M}'\mathfrak{M}'')_*$. Since

$$\mathfrak{F}_{12} = \cap(\mathfrak{F}_1\mathfrak{F}_2),$$

we have

$$\mathfrak{F}_{12} = \mathfrak{F}_1 = \mathfrak{F}_2 = (\mathfrak{M}'\mathfrak{M}'')_*.$$

Theorem.—If a class $\mathfrak{M}$ *of functions of a system* $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$ *has the properties* K_{1*}, K_{2*} *then it has the property* K_{12*}. *In symbols*:

$$\mathfrak{M}^{K_{1*}K_{2*}} \,.\supset.\, \mathfrak{M}^{K_{12*}}.$$

29. *Propositions.*—The following propositions‡ are relative to a class $\mathfrak{P}$, a development Δ of $\mathfrak{P}$ and classes $\mathfrak{M} \equiv [\mu]$, $\mathfrak{N} \equiv [\rho]$, $\mathfrak{S} \equiv [\sigma]$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

(1) $$\mathfrak{M}^{D_1K_1} \,.\, \mathfrak{N}^{K_1\mathfrak{M}} \,.\, \mathfrak{S}^{B_1\mathfrak{N}\mathfrak{M}} \,.\supset.\, \mathfrak{R}_{\mathfrak{S}}{}^{K_1\mathfrak{M}}.$$

(2) $$\mathfrak{M}^{D_1K_1} \,.\, \mathfrak{N}^{K_1\mathfrak{M}\,.\,B_1\mathfrak{N}\mathfrak{M}} \,.\supset.\, \mathfrak{R}_L{}^{K_1\mathfrak{M}\,.\,B_1\mathfrak{N}\mathfrak{M}} \,.\, \mathfrak{R}_*{}^{K_1\mathfrak{M}\,.\,B\mathfrak{N}_1\mathfrak{M}}.$$

(3) $$\mathfrak{M}^{D_1K_1} \,.\supset.\, \mathfrak{M}_L{}^{D_1K_1\mathfrak{M}} \,.\, \mathfrak{M}_{\mathfrak{M}}{}^{D_1K_1\mathfrak{M}} \,.\, \mathfrak{M}_*{}^{D_1K_1\mathfrak{M}}.$$

The proof of (1) is straightforward. (2) and (3) then follow in order by the use of (1) and (2) respectively, and by proposition (5) of § 9*a*. We restate (3) as

Theorem I. If a class $\mathfrak{M}$ *of functions of a system* $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$ *has the dominance property* D_1 *and the property* K_1 *then the linear extension of* $\mathfrak{M}$, *the extension of* $\mathfrak{M}$ *as to itself and the* $*$*-extension of* $\mathfrak{M}$ *each have the dominance property* D_1 *and the property*§ K_1.

* $\mathfrak{F}_i \equiv [\text{all } \varphi^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}]$ $(i = 1, 2, 12)$.

† § 23.

‡ Propositions 1, 2, 3, are to be compared with propositions 5, 6, 7, of § 23*a*.

§ Cf. § 23*a*2$_1$.

The following propositions are relative to a class $\mathfrak{P}'$ with a development Δ', a class $\mathfrak{P}''$, classes $\mathfrak{R} \equiv [\rho]$, $\mathfrak{S} \equiv [\sigma]$ of functions on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$, and classes $\mathfrak{M}'$, $\mathfrak{M}''$ of functions on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively.

$$(4) \qquad \mathfrak{M}'^{D_1 K_1'} . \mathfrak{M}''^{D_1} . \mathfrak{R}^{K_1'\mathfrak{M}'(\mathfrak{M}'')} . \mathfrak{S}^{B_1 \mathfrak{N}\mathfrak{M}'(\mathfrak{M}'')} . \supset . \mathfrak{R}_{\mathfrak{S}}{}^{K_1'\mathfrak{M}'(\mathfrak{M}'')}.$$

$$(5) \quad \mathfrak{M}'^{D_1 K_1'} . \mathfrak{M}''^{D_1} . \mathfrak{R}^{K_1'\mathfrak{M}'(\mathfrak{M}'') . B_1 \mathfrak{N}\mathfrak{M}'\mathfrak{M}''} . \supset . \mathfrak{R}_L{}^{K_1'\mathfrak{M}'(\mathfrak{M}'') . B_1 \mathfrak{N}\mathfrak{M}'\mathfrak{M}''} . \mathfrak{R}_*{}^{K_1'\mathfrak{M}'(\mathfrak{M}'') . B_1 \mathfrak{N}\mathfrak{M}'(\mathfrak{M}'')}.$$

$$(6) \quad \mathfrak{M}'^{D_1 K_1'} . \mathfrak{M}''^{D_1} . \supset . (\mathfrak{M}'\mathfrak{M}'')_L{}^{D_1 K_1'\mathfrak{M}'(\mathfrak{M}'')} . (\mathfrak{M}'\mathfrak{M}'')_{(\mathfrak{M}'\mathfrak{M}'')}{}^{D_1 K_1'\mathfrak{M}'(\mathfrak{M}'')} . (\mathfrak{M}'\mathfrak{M}'')_*{}^{D_1 K_1'\mathfrak{M}'(\mathfrak{M}'')}.$$

(4, 5, 6) are generalizations in the sense of relativity of (1, 2, 3) and the proof follows the same lines. We state (6) as

THEOREM II. *If a class $\mathfrak{M}'$ of functions on $\mathfrak{P}'$ has the property D_1 and the property K_1' relative to a development Δ', and if any second class $\mathfrak{M}''$ of functions on $\mathfrak{P}''$ has the property D_1; then the linear extension of the composite class $\mathfrak{M}'\mathfrak{M}''$, the extension of $\mathfrak{M}'\mathfrak{M}''$ as to itself, and the $*$-extension of $\mathfrak{M}'\mathfrak{M}''$ have the properties D_1, $K_1'\mathfrak{M}'(\mathfrak{M}'')$.*

The following propositions are generalizations in the sense of relativity of propositions (5_2, 6_2, 7_2) of § 23*a*.

$$(7) \qquad \mathfrak{M}'^{D} . \mathfrak{M}''^{D} . \mathfrak{R}^{K_2'\mathfrak{M}'(\mathfrak{M}'')} . \mathfrak{S}^{B_1 \mathfrak{N}\mathfrak{M}'(\mathfrak{M}'')} . \supset . \mathfrak{R}_{\mathfrak{S}}{}^{K_2'\mathfrak{M}'(\mathfrak{M}'')}.$$

$$(8) \quad \mathfrak{M}'^{D} . \mathfrak{M}''^{D} . \mathfrak{R}^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_1 \mathfrak{N}\mathfrak{M}'\mathfrak{M}''} . \supset . \mathfrak{R}_L{}^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_1 \mathfrak{N}\mathfrak{M}'\mathfrak{M}''} . \mathfrak{R}_*{}^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_1 \mathfrak{N}\mathfrak{M}'\mathfrak{M}''}.$$

$$(9) \quad \mathfrak{M}'^{DK_2'} . \mathfrak{M}''^{D} . \supset . (\mathfrak{M}'\mathfrak{M}'')_L{}^{DK_2'\mathfrak{M}'(\mathfrak{M}'')} . (\mathfrak{M}'\mathfrak{M}'')_{\mathfrak{M}'\mathfrak{M}''}{}^{DK_2'\mathfrak{M}'(\mathfrak{M}'')} . (\mathfrak{M}'\mathfrak{M}'')_*{}^{DK_2'\mathfrak{M}'(\mathfrak{M}'')}.$$

THEOREM III. *If a class $\mathfrak{M}'$ of functions on $\mathfrak{P}'$ has the property D and the property K_2' relative to a development Δ' and if any second class $\mathfrak{M}''$ of functions on $\mathfrak{P}''$ has the property D, then the classes $(\mathfrak{M}'\mathfrak{M}'')_L$, $(\mathfrak{M}'\mathfrak{M}'')_{\mathfrak{M}'\mathfrak{M}''}$, $(\mathfrak{M}'\mathfrak{M}'')_*$ each possess the properties D, $K_2'\mathfrak{M}'(\mathfrak{M}'')$.*

30. *Propositions.*—Relative to a class $\mathfrak{P}'$ with development Δ' and a class $\mathfrak{M}'$ of functions on $\mathfrak{P}'$, we have the propositions:

$$(1) \qquad \mathfrak{M}' \ni \left(\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_*{}^{K_2'\mathfrak{M}'(\mathfrak{M}'')}\right) :\supset : \mathfrak{M}'^{K_1'_*} . \supset . \mathfrak{M}'^{K_{12}'_*},$$

viz., if $\mathfrak{M}'$ is such that for every $\mathfrak{M}''^{LCD}$ (it is true that) $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$ then if $\mathfrak{M}'$ has the property K_{1*}' it also has the property K_{12*}'.

$$(2) \qquad \mathfrak{M}' \ni \left(\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_*{}^{K_1'\mathfrak{M}'(\mathfrak{M}'')}\right) :\supset : \mathfrak{M}'^{K_2'_*} . \supset . \mathfrak{M}'^{K_{12}'_*}.$$

$$(3) \qquad \mathfrak{M}'^{K_1'_* K_{12}'_*} . \supset . \mathfrak{M}' \ni \left(\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_*{}^{K_2'\mathfrak{M}'(\mathfrak{M}'')}\right),$$

viz., if $\mathfrak{M}'$ has the properties K'_{1*}, K'_{12*} then $\mathfrak{M}'$ is such that for any $\mathfrak{M}''^{LCD}$ it is so that $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K'_2\mathfrak{M}'(\mathfrak{M}'')$.

(4) $$\mathfrak{M}'^{K_{2'*}K_{12'*}} . \supset . \mathfrak{M}' \ni \left(\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_*^{K_1'\mathfrak{M}'(\mathfrak{M}'')}\right).$$

(5) $$\mathfrak{M}'^{K_{1'*}} . \supset . \mathfrak{M}' \ni \left(\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_*^{K_2'\mathfrak{M}'(\mathfrak{M}'')} : \infty : \mathfrak{M}'^{K_{12'*}}\right).$$

(6) $$\mathfrak{M}'^{K_{2'*}} . \supset . \mathfrak{M}' \ni \left(\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_*^{K_1'\mathfrak{M}'(\mathfrak{M}'')} : \infty : \mathfrak{M}'^{K_{12'*}}\right).$$

(7) $$\mathfrak{M}'^{D_1K_1'K_2'*} . \supset . \mathfrak{M}'^{K_{12'*}}.$$

(8) $$\mathfrak{M}'^{DK_2'K_1'*} . \supset . \mathfrak{M}'^{K_{12'*}}.$$

(1, 2, 3, 4) follow from direct consideration of the definitions of the properties K'_{1*}, K'_{2*}, K'_{12*} and the classes of functions involved in these definitions. (5) and (6) follow from (1, 3) and (2, 4) respectively. (7) follows from (2) and proposition (6) of § 29. (8) follows from (1) and proposition (9) of § 29. We state (5, 6) and (7, 8) respectively as the following theorems:

THEOREM I. *If a class $\mathfrak{M}'$ of functions on $\mathfrak{P}'$ has the property $\frac{K_1'*}{K_2'*}$ relative to a development Δ', then the statement that $(\mathfrak{M}'\mathfrak{M}'')_*$ ($\mathfrak{M}''$ an arbitrary class of functions on $\mathfrak{P}''$ possessing the properties LCD) has the property $\frac{K_2'\mathfrak{M}'(\mathfrak{M}'')}{K_1'\mathfrak{M}'(\mathfrak{M}'')}$ is equivalent to the statement that $\mathfrak{M}'$ has the property K'_{12*}.*

THEOREM II. *If a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ has the property $\frac{D_1}{D}$, and the properties $\frac{K_1, K_2*}{K_2, K_1*}$ relative to a development Δ, then the class $\mathfrak{M}$ has the property K_{12*}.*

31. It is not difficult to see that whatever the class $\mathfrak{P}$ and its development Δ there exists a developmental system $\mathfrak{D} \equiv ((\delta^{ml}))$ such that conditions 1*a* and 1′*b* of § 17 are holding. Furthermore, it is clear that the class $\mathfrak{D}$ is bounded by the class of constants. These considerations give rise to the following propositions relative to a class $\mathfrak{P}$ and a development Δ:

(1) $$\mathfrak{M} \ni [\text{all } \mu^{B_1\mathfrak{M}(\mathfrak{P})}]^{B_0\mathfrak{M}} . \supset . \mathfrak{M}^\Delta,$$

viz., if $\mathfrak{M}$ is such that all functions μ, which are dominated by the class of constants on $\mathfrak{P}$, belong to $\mathfrak{M}$ then $\mathfrak{M}$ has the property Δ.

The following propositions then follow in order:

(2) $$\mathfrak{N} . \mathfrak{M} = [\text{all } \mu^{K_1\mathfrak{N}}] = [\text{all } \mu^{K_2\mathfrak{N}}] . \supset . [\text{all } \mu^{B_1\mathfrak{N}(\mathfrak{P})}]^{B_0\mathfrak{M}} . \mathfrak{M}^\Delta.$$

(3 $$\mathfrak{M}^{K_{1*}K_{2*}} . \supset . [\text{all } \mu^{B_1\mathfrak{M}(\mathfrak{P})}]^{B_0\mathfrak{M}_*} . \mathfrak{M}_*{}^\Delta.$$

(4) $$\mathfrak{M}^{LCK_{1*}K_{2*}} . \supset . [\text{all } \mu^{B_1\mathfrak{M}(\mathfrak{P})}]^{B_0\mathfrak{M}} . \mathfrak{M}^\Delta.$$

(2) follows from §26.1; §23$a3_1$; §31.1. (3) follows from §23$a11_i$; § 31.2. (4) is a corollary of (3). We state (3) as

THEOREM I. *If a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ has the properties K_{1*}, K_{2*} relative to the development Δ then the $*$-extension of $\mathfrak{M}$ contains the class of constants and possesses the property Δ.*

32. *Propositions.*—The following proposition relative to two functions μ, σ on $\mathfrak{P}$ with development Δ is fundamental and is to be compared with the proposition, "every function continuous on a closed interval is bounded," in the ordinary theory of continuous functions.

(1) $$\Delta^{C_2} . \mu^{K_2\sigma} . \supset . \mu^{B_1\mathfrak{N}(\mathfrak{P})},$$

viz., if the development Δ has the property C_2 and μ has the property $K_2\sigma$ then μ is dominated by the class of constants on $\mathfrak{P}$.

Proof.—

$$\mu^{K_2\sigma} . \equiv . \mu \ni \left(e : \supset : \exists m_e \ni K_{p_1p_2m_e} . \supset . A(\mu_{p_1} - \mu_{p_2}) \leqq eA\sigma_{p_1}\right).$$

For $e = 1$ we have

$$K_{p_1p_2m_1} . \supset . A\mu_{p_2} \leqq A\sigma_{p_1} + A\mu_{p_1}.$$

Since Δ has the property C_2, there exists a stage $m_0 \geqq m_1$, of the development which is complete. Also there exists a finite number of elements p_1 such that each element p_2 is directly connected* with some one of the elements p_1 at the stage m_0. Therefore there exists a function constant as to $\mathfrak{P}$ which dominates μ. Proposition (2) follows at once.

(2) $$\Delta^{C_2} . \mathfrak{M}^{K_2} . \supset . \mathfrak{M}^{B_1\mathfrak{N}(\mathfrak{P})}.$$

In the following proposition relative to a class $\mathfrak{P}$, with a development Δ conditioned as indicated, the system $\mathfrak{D}(\mathfrak{M}) \equiv ((\delta^{ml}))$ is understood to be the system whose existence is implied by the fact that $\mathfrak{M}$ has the property Δ.

(3) $$\Delta^{C_2} . \mathfrak{D}(\mathfrak{M}) . \mathfrak{M}^{\Delta B_1 A^{\mathfrak{P}}} . : \supset :. m_0 . e : \supset : \exists m_e > m_0 \ni p . \supset . \sum_h A\delta_p^{m_e h} \leqq e.$$

Proof.—By condition (1b) of § 17 on the developmental system and $\mathfrak{M}^{\mathfrak{B}_1\mathfrak{A}}$ we have

$$\mu . e : \supset : \exists m_{\mu e} \ni (m \geqq m_{\mu e} . p) . \supset . \sum_h A\mu_{r^{mh}} \delta_p^{mh} \leqq e.$$

Recognizing that this holds in particular if μ is any one of the func-

* Cf. § 14.

tions of the system $((\delta^{ml}))$ we have, denoting by n an arbitrary stage and by ns the corresponding bipartite index,

$$n \,.\, e : \supset : \exists\, m_{ne} \ni (m \geqq m_{ne} \,.\, p) \,.\supset.\, \sum_s \sum_h A\delta^{ns}_{r^{mh}}\, \delta^{mh}_p \leqq e;$$

$$n \,.\, e/2 : \supset : \exists\, m_{n\,e/2} \ni (m \geqq m_{n\,e/2} \,.\, p) \,.\supset.\, \sum_h A\delta^{mh}_{p\gamma} \sum_s A\delta^{ns}_{r^{mh}} \leqq e/2.$$

By condition 1*a*, § 17, and the hypothesis Δ^{C_2}, there exists a stage n_0 which is complete and such that

$$p \,.\supset.\, \sum_y A\delta^{n_0 q}_p \geqq 1/2.$$

Use this n_0 as n in above and then any m exceeding both $m_{n_0 e}$ and m_0 such that $\mathfrak{P}^{m-V}$ is effective* as the m_e desired in the theorem.

(4) $$\Delta^{C_2} \,.\, \mathfrak{M}^{L\Delta \cdot B_1 \mathfrak{P}_1} \,.\supset.\, \exists\, \mu^{B_0 \mathfrak{M}} \,.\text{ uniformly bounded from } 0 \,.\dagger$$

Proof.—By condition 1*a*, § 17, on developmental systems

$$e : \supset : \exists\, m'_e \ni (m \geqq m'_e \,.\, p) \,.\supset.\, A(\sum_g \delta^{mg}_p - 1) \leqq e.$$

By (3)

$$e \,.\, m'_e : \supset : \exists\, m_1 > m'_e \ni p \,.\supset.\, \sum_h A\delta^{m_1 h}_p \leqq e.$$

For the stage m_1 we have

$$1 - 2e \leqq \sum_l \delta^{m_1 l}_p \leqq 1 + 2e.$$

Take $e = 1/4$ and the function $\sum_l \delta^{m_1 l}_p$ is the function desired.

Proposition (5) relative to a general class $\mathfrak{P}$ is readily proved.

(5) $$\mathfrak{M}^{B_1 \mathfrak{P}_1 \cdot C} \ni \exists\, \mu^{B_0 \mathfrak{M}} \,.\text{ uniformly bounded from } 0 \,.\supset.\, \mathfrak{M}^D.$$

Through (1 . . . 5) we are able to prove the following fundamental propositions relative to a class $\mathfrak{P}$ with a development Δ conditioned as indicated.

(6) $$\Delta^{C_2} \,.\, \mathfrak{M}^{L\Delta K_2} \,.\supset.\, \mathfrak{M}^D.$$

(7) $$\Delta^{C_2} \,.\, \mathfrak{M}^{L\Delta K_1 K_2} \,.\supset.\, \mathfrak{M}^{DK_{12*}} \,.\, \mathfrak{M}_*{}^{LCDA\Delta K_{12} K_{12*} \cdot B_1 \mathfrak{P}_1 \mathfrak{M}}.$$

(8) $$\Delta^{C_2} \,.\, \mathfrak{M}^{LC\Delta K_1 K_2} \,.\supset.\, \mathfrak{M}^{DK_{12*}A}.$$

(9) $$\Delta^{C_1} \,.\, \mathfrak{M}^{L\Delta K_2} \,.\supset.\, \mathfrak{M}^{DK_{2*}K_{12*}} \,.\, \mathfrak{M}_*{}^{LCDA\Delta K_{12} K_{2*} K_{12*} \cdot B_1 \mathfrak{P}_1 \mathfrak{M}}.$$

(10) $$\Delta^{C_1} \,.\, \mathfrak{M}^{LC\Delta K_2} \,.\supset.\, \mathfrak{M}^{DK_{2*}K_{12*}A}.$$

* The necessity of choosing a non-vacuous stage should be noticed. In case a stage is vacuous any functions at all are effective as the developmental functions of that stage.

† $\mu^{\text{uniformly bounded from } 0} \,.\equiv.\, \mu \ni (\exists\, e \ni A\mu_p \geqq e \;\; (p))$.

(6) follows from (1, 2, 3, 4, 5). (7) follows from (6) and from § 23*a*17. (8) is a corollary of (7). (9) follows from (7) and § 16*a*2; § 20*a*1; § 27.11. (10) is a corollary of (9). We state (7) as the following theorem:

THEOREM I. *If a class* $\mathfrak{M}$ *of functions on* $\mathfrak{P}$, *with a development* Δ *having the property* C_2, *has the properties*:

$$L;\ \Delta;\ K_1;\ K_2,$$

then the class $\mathfrak{M}$ *has also the properties*:

$$D;\ K_{12*},$$

and the class $\mathfrak{M}_*$ *has the properties*:

$$L;\ C;\ D;\ A;\ \Delta;\ K_1;\ K_2;\ K_{12*};\ B_1\mathfrak{A}\mathfrak{M}.$$

Here under the condition that the development Δ has the property C_2 we secure the properties D; K_{12*} as a result of the properties L; Δ; K_1; K_2. This is to be compared with Theorem II of § 82 of The Introduction to General Analysis* where it is shown that for a general development Δ the property K_{12*} is implied by the properties D; Δ; K_1; K_2. It will be seen that the theorem which we give is applicable in many important cases. The ordinary development of the linear interval (01), for instance, has the property C and therefore the properties C_1 and C_2.

33. Relative to classes $\mathfrak{P}'$; $\mathfrak{P}''$ with developments Δ'; Δ'' and the composite class $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ with the composite development† Δ'' we have the following proposition:

$$(1)\qquad \Delta'^{C_2} . \Delta''^{C_2} . \mathfrak{M}'^{L\Delta' K_1' K_2'} . \mathfrak{M}''^{L\Delta'' K_1'' K_2''} . \supset . (\mathfrak{M}'\mathfrak{M}'')^{D\Delta'' K_{12} K_{12*}} (\mathfrak{M}'\mathfrak{M}'')_*^{LCDA\Delta'' K_{12} K_{12*} B_1 \mathfrak{A}\mathfrak{M}'\mathfrak{M}''}.$$

This follows from § 32.6; § 12*a*2; § 21*a*1, 2, 3, 4; and § 23*a*17.

The following proposition is a corollary of (1) through § 16*a*2 and § 20*a*1; § 27.11.

$$(2)\qquad \Delta'^{C_1} . \Delta''^{C_1} . \mathfrak{M}'^{L\Delta' K_2'} . \mathfrak{M}''^{L\Delta'' K_2''} . \supset . (\mathfrak{M}'\mathfrak{M}'')^{D\Delta'' K_{12} K_{2*} K_{12*}} . (\mathfrak{M}'\mathfrak{M}'')_*^{LCDA\Delta'' K_{12} K_{2*} K_{12*} . B_1 \mathfrak{A}\mathfrak{M}'\mathfrak{M}''}.$$

By (2) and § 21*a*9 we have the following theorem:

* Cf. § 23*a*17.

† § 21.

THEOREM I. *In the case of two systems*:

$$(\mathfrak{A};\ \mathfrak{P}';\ \Delta'^{C_1};\ \mathfrak{M}'^{P'});\ (\mathfrak{A};\ \mathfrak{P}'';\ \Delta''^{C_1};\ \mathfrak{M}''^{P''}),$$

where

$$P' \equiv L\Delta' K_2'; \qquad P'' \equiv L\Delta'' K_2'',$$

the composite systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta'';\ \mathfrak{M}'\mathfrak{M}'');\quad (\mathfrak{A};\ \mathfrak{P};\ \Delta'';\ (\mathfrak{M}'\mathfrak{M}'')_*),$$

are such that

$$\Delta''^{C_1} \cdot (\mathfrak{M}'\mathfrak{M}'')^{D\Delta'' K_{12} K_{12*}} \cdot (\mathfrak{M}'\mathfrak{M}'')_*^{LCDA\Delta'' K_{12} K_{12*} \cdot B_1^{\mathfrak{M}'\mathfrak{M}''}}.$$

From Theorem I and § 32.9 we secure the following theorem:

THEOREM II. *The genus of all systems*:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta^{C_1};\ \mathfrak{M}^P),$$

where

$$P \equiv L(D)\Delta(K_1)K_2(K_{2*}K_{12*}),$$

is closed under the operations A, C:

A: **-extension of classes* $\mathfrak{M}$;

C: *simultaneous composition of classes* $\mathfrak{P}$ *and developments* Δ *and *-composition of classes* $\mathfrak{M}$;

and combinations of these operations.

The subgenus of all systems obtained by the operations A, C and their combinations is the genus of all systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta^{C_1};\ \mathfrak{M}^{P_0}),$$

where

$$P_0 \equiv LC(DA)\Delta(K_1)K_2(K_{2*}K_{12*}).$$

This genus is closed under the operations A, C and their combinations. To this genus belong the systems:

$$(\mathfrak{A};\ \mathfrak{P}^{\mathrm{I}};\ \Delta^{\mathrm{I}};\ \mathfrak{M}^{\mathrm{I}});\qquad (\mathfrak{A};\ \mathfrak{P}^{\mathrm{II}n};\ \Delta^{\mathrm{II}n};\ \mathfrak{M}^{\mathrm{II}n});\qquad (\mathfrak{A};\ \mathfrak{P}^{\mathrm{IV}};\ \Delta^{\mathrm{IV}};\ \mathfrak{M}^{\mathrm{IV}}).$$

Here the properties occurring in parentheses are implied by the other properties of the composite property in which they occur in view of Δ^{C_1} and § 32.6, 9, 10; § 27.11; § 20*a*1.

34. *Propositions.*—

$$(1) \qquad \mathfrak{P} \cdot p_1 \cdot \Delta^{V_2(p_1)} \cdot \mu^{K_1\sigma} \cdot \supset \cdot \mu_{p_1} = 0,$$

viz., if the development Δ of $\mathfrak{P}$ has the property* V_2 as to p_1 a particular

* $\Delta^{V_2(p)} \cdot \equiv \cdot \Delta \ni (m \cdot \supset \cdot \exists m_0 > m \ni p^{-m_0})$. Cf. § 22.

element of $\mathfrak{P}$ and if any function μ on $\mathfrak{P}$ has the property $K_1\sigma$ where σ is any second function on $\mathfrak{P}$ then $\mu_{p_1} = 0$.

We define a reduced class $\mathfrak{P}_R$ of $\mathfrak{P}$,

$$\mathfrak{P}_R \equiv \mathfrak{P} - \mathfrak{S}$$

where $\mathfrak{S}$ is a subclass of $\mathfrak{P}$ and

$$s^{B_0 \xi} . \supset . \Delta^{V_2(s)}.$$

Relative to this situation we have the following propositions:

(2) $$\mathfrak{M}^{K_1} . \mu . s . \supset . \mu_s = 0;$$

(3) $$\mathfrak{M}^{K_1} : \supset : \mathfrak{M}^{A} . \sim . \mathfrak{M}_R{}^{A} . \mathfrak{M}^{C} . \sim . \mathfrak{M}_R{}^{C} . \mathfrak{M}^{D} . \sim . \mathfrak{M}_R{}^{D} . \mathfrak{M}^{D_1} . \sim . \mathfrak{M}_R{}^{D_1} . \mathfrak{M}^{L} . \sim . \mathfrak{M}_R{}^{L};$$

(4)* $$\mathfrak{M}^{K_1 \Delta} . \supset . \Delta^{V_1(\mathfrak{S})}.$$

Consider also a development Δ_R of $\mathfrak{P}_R$, defined as in § 22. We have the following propositions as to properties Δ; K_1; K_2; K_{1*}; K_{2*}; K_{12*} of $\mathfrak{M}$ relative to Δ and the same properties of $\mathfrak{M}_R$ relative to Δ_R.

(5) $$\mathfrak{M}^{K_1} . \supset . \mathfrak{M}_R{}^{K_1}.$$

(6) $$\mathfrak{M}^{K_1} : \supset : \mathfrak{M}^{\Delta} . \sim . \mathfrak{M}_R{}^{\Delta}.$$

(7)† $$\mathfrak{M}^{K_1} . \exists \mathfrak{S} . \supset . \mathfrak{M}^{-K_{2*}}.$$

(8) $$\mathfrak{M}^{K_1 \Delta} : \supset : \mathfrak{M}^{K_2} . \sim . \mathfrak{M}_R{}^{K_2} . \mathfrak{M}^{K_{1*}} . \sim . \mathfrak{M}_R{}^{K_{1*}} . \mathfrak{M}^{K_{12*}} . \sim . \mathfrak{M}_R{}^{K_{12*}}.$$

Propositions (9) and (10) are relative to a class $\mathfrak{P} = \mathfrak{R} + \mathfrak{S}$.

(9)‡ $$\Delta^{C_2 \backslash V_2 \xi} . \mathfrak{M}^{L \Delta K_1 K_2} . \supset . \mathfrak{M}^{D K_{12*}} . \mathfrak{M}_*{}^{L C D_1 \Delta K_{12} K_{12*} . B_1 \mathfrak{M}}.$$

(10) $$\Delta^{C_2 \backslash V_2 \xi} . \mathfrak{M}^{L C \Delta K_1 K_2} . \supset . \mathfrak{M}^{D K_{12*} A}.$$

(11)§ $$\Delta^{C_2 \backslash} . \mu^{K_2 \sigma} . \supset . (\mu_r \mid r^{\mathfrak{R}})^{B_1 \mathfrak{M} \backslash}.$$

35. Through the propositions § 23*a*16; § 29.6; § 27.10; § 20*a*2′; § 20*a*2; § 29.5; § 18*a*2, 3; § 29I; § 9*a*5 we are led to the following propositions:

(1) $$\mathfrak{P}' . \Delta'^{F_1} . \mathfrak{M}'^{D_1 \Delta' K_1'} . \mathfrak{P}'' . \mathfrak{M}''^{L C D_1} . \supset . (\mathfrak{M}' \mathfrak{M}'')_* = [\text{all } \varphi^{K_1' \mathfrak{M}' \mathfrak{M}'' . B_0 \mathfrak{M}'' p'}] = [\text{all } \varphi^{K_{12}' \mathfrak{M}' \mathfrak{M}'' . B_0 \mathfrak{M}'' p'}].$$

(2) $$\mathfrak{P}' . \Delta'^{F_1} . \mathfrak{M}'^{D_1 \Delta' K_1'} . \supset . \mathfrak{M}'^{K_2' K_1' * K_{12}' *} . \mathfrak{M}_L'^{D_1 \Delta' K_{12}' K_1' * K_{12}' *} . \mathfrak{M}_*'^{L D_1 \Delta' K_{12}' K_1' * K_{12}' *}.$$

* $\Delta^{r_1 \xi} . \equiv . \Delta \ni (\exists m_0 \ni m > m_0 . \supset . \mathfrak{S}^{-m})$. Cf. § 22.

† That is, in case the class $\mathfrak{S}$ is really an existent class, the class $\mathfrak{M}$ cannot have the property K_{2*}.

‡ Cf. § 32.7.

§ Cf. § 32.1.

THEOREM I. *If a class* $\mathfrak{M}'$ *of functions of a system* $(\mathfrak{A}; \mathfrak{P}'; \Delta'^{F_1}; \mathfrak{M}')$ *has the properties* D_1, Δ', K_1' *then*

(a)
$$\mathfrak{P}'' \,.\, \mathfrak{M}''^{LCD_1} \,.\supset.\, (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \varphi^{K_{12}'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}]$$
$$= [\text{all } \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}];$$

(b) $\mathfrak{M}'$ *has the properties* K_2', K_{1*}', K_{12*}';

(c) $\mathfrak{M}_L'$ *and* $\mathfrak{M}_*'$ *each has the properties* L, D_1, Δ', K_{12}', K_{1*}', K_{12*}'.

Theorem I*a* furnishes, under special hypotheses, a functional characterization of $(\mathfrak{M}'\mathfrak{M}'')_*$ where $\mathfrak{M}''$ is subject to conditions less exacting than the *LCD* used in the definitions of K_{1*}', K_{2*}', K_{12*}'.*

In proposition (2) under the hypothesis Δ^{F_1} we secure K_{1*}, K_{12*} as a result of D_1, Δ, K_1. This is to be compared with Theorem II of § 82 of the Introduction to General Analysis.†

THEOREM II. *In the case of two systems*:

$$(\mathfrak{A}; \mathfrak{P}'; \Delta'^{F_1}; \mathfrak{M}'^{P'}); (\mathfrak{A}; \mathfrak{P}''; \Delta''^{F_1}; \mathfrak{M}''^{P''}),$$

where

$$P' \equiv D_1\Delta'K_1'; \qquad P'' \equiv D_1\Delta''K_1'',$$

the composite systems:

$$(\mathfrak{A}; \mathfrak{P}; \Delta'''; \mathfrak{M}'\mathfrak{M}''); \quad (\mathfrak{A}; \mathfrak{P}; \Delta'''; (\mathfrak{M}'\mathfrak{M}'')_L); \quad (\mathfrak{A}; \mathfrak{P}; \Delta'''; (\mathfrak{M}'\mathfrak{M}'')_*)$$

are such that

$$\Delta'''^{F_1} \,.\, (\mathfrak{M}'\mathfrak{M}'')^{D_1\Delta'''K_{12}K_{1*}K_{12*}} \,.\, (\mathfrak{M}'\mathfrak{M}'')_L^{D_1\Delta'''K_{12}K_{1*}K_{12*}} \,.\, (\mathfrak{M}'\mathfrak{M}'')_*^{LD_1\Delta'''K_{12}K_{1*}K_{12*}}.$$

This theorem follows from Theorem I and § 21*a*1, 3, 7; § 20*a*2.

THEOREM III. *The genus of all systems*:

$$(\mathfrak{A}; \mathfrak{P}; \Delta^{F_1}; \mathfrak{M}^P)$$

where

$$P \equiv D_1\Delta K_1(K_2)K_{1*}(K_{12*}),$$

is closed under the five operations A, B, C_0, C_1, C_2:

A: *linear extension of classes* $\mathfrak{M}$;

B: $*$*-extension of classes* $\mathfrak{M}$;

C: *simultaneous composition of classes* $\mathfrak{P}$ *and developments* Δ *and*

C_0: *multiplication of classes* $\mathfrak{M}$;

C_1: *linear composition of classes* $\mathfrak{M}$;

C_2: $*$*-composition of classes* $\mathfrak{M}$,

and the combinations of these operations.

* Cf. I. G. A., § 65*d*II; § 67.

† § 23*a*17.

The subgenus of all systems obtained by operations A, B, C_1, C_2 and their combinations is the genus:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta^{F_1};\ \mathfrak{M}^{P_0}),$$

where

$$P_0 \equiv LD_1\Delta K_1(K_2)K_{1*}(K_{12*}).$$

This genus is closed under the operations A, B, C_1, C_2 and their combinations. To this genus belong the systems:

$$\mathrm{I};\ \mathrm{II}_n;\ \mathrm{III};\ \mathrm{III}_e;\ \mathrm{III}_0,$$

in each case the development of the system being the ordinary development of the corresponding class $\mathfrak{P}$.

Here the properties in parenthesis are implied by the other constituent properties of the respective composite properties in view of the hypothesis Δ^{F_1} and propositions § 27.10; § 20*a*2.

36. The theorem of this section is to be compared with Theorem III, § 82, of the Introduction to General Analysis.* It follows readily from § 23*a*15, 1_{12}.

THEOREM I. *From*

$$\mathfrak{P};\ \Delta;\ \mathfrak{M}^{LCD_1\Delta K_{12}}$$

it follows that

$$\mathfrak{M}^A \,.\, \mathfrak{M} = [all\ \varphi^{K_{12}\text{``}}] \,.\, (B_0\mathfrak{M} \sim K_{12}\mathfrak{M}).$$

37. *Notation and definition.*—A function μ on $\mathfrak{P}$ with development Δ is said to be ultimately zero as to the development, in notation:

$$\mu^{\mathrm{ult}\ 0\ \Delta},$$

in case there exists a stage m of the development such that if p is undeveloped for any stage $m_0 \geqq m$ then $\mu_p = 0$. That is,

$$\mu^{\mathrm{ult}\ 0,\Delta} \,.\equiv.\, \mu \ni (\exists\, m \ni K_{pm} \,.\supset.\, \mu_p = 0).$$

It is to be noticed that in case of $\mathfrak{P}^{\mathrm{III}}$ with ordinary development this definition of ultimately zero coincides with the ordinary notion of ultimately zero.

38. *Propositions.*—

(1) $$\mathfrak{P} \,.\, \Delta \,.\, \varphi^{\mathrm{ult}\ 0(\Delta)} \,.\, \mathfrak{M} \,.\supset.\, \varphi^{K_1\text{``}}.$$

(2) $$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{K_{1*}} \,.\supset.\, [\mathrm{all}\ \mu^{\mathrm{ult}\ 0\ \Delta}]^{B_0\mathfrak{M}_*}.$$

(3) $$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{LCK_{1*}} \,.\supset.\, [\mathrm{all}\ \mu^{\mathrm{ult}\ 0\ \Delta}]^{B_1\text{``}\mathfrak{M}}.$$

* § 23*a*18.

(2) follows from (1) and § 23a11$_1$. (3) is a corollary of (2).

(4) $$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{LCK_1} \ni [\text{all } \mu^{\text{ult } 0(\Delta)}]^{B_0\mathfrak{M}} \,.\, \supset \,.\, \mathfrak{M}^{A}.$$

Proof.—Under the hypotheses on $\mathfrak{M}$ we have

(a) $\mu \,.:\supset:. \exists \mu_0 \ni e :\supset: \exists m_e \ni K_{pm_e} \,.\, \supset \,.\, A\mu \leqq eA\mu_0$,

(b) $\mu \,.\, m :\supset: \exists \mu_{\mu m} \ni (K_{pm} \,.\, \supset \,.\, \mu_{\mu mp} = 0) \,.\, (^{-}K_{pm} \,.\, \supset \,.\, \mu_{\mu mp} = A\mu_p)$,

(c) $\underset{m}{\mathbf{L}}\, \mu_{\mu m} = A\mu \;\; (\mathfrak{P}; \mu_0)$,

(d) $\mu^{B_0\mathfrak{M}} \,.\, \supset \,.\, A\mu^{B_0\mathfrak{M}}$.

(5) $\mathfrak{P} \,.\, \Delta^{F_1} \,.\, \mathfrak{M}^{LC\Delta K_1} \ni [\text{all } \mu^{\text{ult } 0(\Delta)}]^{B_0\mathfrak{M}} \,.\, \supset \,.\, \mathfrak{M}^{AK_{1*}K_{12*}}$.

(6) $\mathfrak{P} \,.\, \Delta^{F_1} \,.\, \mathfrak{M}^{LC\Delta K_1} :\supset: \mathfrak{M}^{A} \,.\, \infty \,.\, \mathfrak{M}^{D_1} \,.\, \infty \,.\, \mathfrak{M}^{K_{1*}} \,.\, \infty \,.\, \mathfrak{M}^{K_{12*}} \,.\, \infty \,.\, [\text{all } \mu^{\text{ult } 0(\Delta)}]^{B_0\mathfrak{M}}$.

(5) follows from (4) and § 7.2; § 35Ib; § 27.10. (6) follows from (2, 3, 4, 5); § 7.2; § 35Ib; § 27.10; § 36I; § 20a2.

(7) $$\mathfrak{P} \,.\, \Delta^{F_1} \,.\, \mathfrak{M}^{L\Delta} \,.\, p_1 \ni \Delta^{C_1(p_1)} \,.\, \supset \,.\, \exists\, \xi^{B_0\mathfrak{M}_*}_{p_1} \ni \xi_{p_1p} = 1 \;\; (p = p_1) \,.\, \xi_{p_1p} = 0 \;\; (p \neq p_1).$$

Proof.—Considering the conditions (1a), (1b) of § 17 on the developmental system it is not difficult under the conditions of this theorem to show that there exists a sequence $\{u_m\}$ of functions of $\mathfrak{M}$ satisfying the following conditions:

(a) $e :\supset: \exists m_e \ni m \geqq m_e \,.\, \supset \,.\, A(\mu_{mp_1} - 1) \leqq e$,

(b) $\mu \,.:\supset:. \exists \mu_0 \ni e :\supset: \exists m_e \ni (m \geqq m_e \,.\, p \neq p_1) \,.\, \supset \,.\, A\mu_{p_1}\mu_{mp} \leqq eA\mu_{0p}$.

It is not difficult to see that μ may be taken such that $\mu_{p_1} = 1$, and that there is available a scale function μ_0 such that $\mu_{0p_1} = 1$.*

(c) $\underset{m}{\mathbf{L}}\, \mu_m = \xi_{p_1} \;\; (\mathfrak{P}; \mu_0)$.

(8) $\mathfrak{P} \,.\, \Delta^{F_1} \,.\, \mathfrak{M}^{LC\Delta} \,.\, \supset \,.\, [\text{all } \mu^{\text{ult } 0(\Delta)}]^{B_0\mathfrak{M}}$.

From (6) and (8) we get

THEOREM I. *From*

$$\mathfrak{P};\ \Delta^{F_1};\ \mathfrak{M}^{LC\Delta K_1}$$

it follows that

$$\mathfrak{M}^{AD_1K_{1*}K_{12*}}.\ [\text{all } \mu^{\text{ult } 0(\Delta)}]^{B_0\mathfrak{M}}.$$

* This readily follows on account of $\mathfrak{M}^L$ unless $\mu_{0p_1} = 0$. In this case a $\bar{\mu}_0$, such that $\bar{\mu}_{0p_1} \neq 0$, available as scale function in (b), may be secured by adding to μ_0 a proper function from the sequence $\{\mu_m\}$.

We notice that Δ^{F_1} implies the existence of a stage m_0 such that for any succeeding stage m the class $\mathfrak{P}^m$ contains only a finite number of elements and the class $\cup[\mathfrak{P}^m, m > m_0]$ is denumberable. Recalling § 34 we see that propositions 4, 5, 6, of the present section and Theorem I are to be applied especially to $\mathfrak{P}^{III}$.

39. *Notations and definitions.*—In § 14 are given the definitions of the classes $\mathfrak{P}^m$ $\mathfrak{P}^{-m}$, $\mathfrak{P}_m$, etc. In this section we define other subclasses of the class $\mathfrak{P}$, specify notations, and define certain properties of functions and classes of functions. These definitions and notations are to be used in the immediate sequel. The first two definitions are relative to a class $\mathfrak{P}$, the others relative to a class $\mathfrak{P}$ with a development Δ.

(1)* $$\mathfrak{M}^{B_2} . \equiv . \mathfrak{M} \ni \big(\exists (\mu . e) \ni p . \supset . A\mu_p \geqq e \big).$$

(2)† $$\mu^{B_2} . \equiv . \mu \ni (\exists e \ni p . \supset . A\mu_p \geqq e).$$

(3)‡ $$\mathfrak{P}_{-m} . \equiv . [p_{-m}] \equiv \mathfrak{P} - \mathfrak{P}_m.$$

(4) $$K_{pm} . \equiv . (p, m) \ni p^{B_0 \mathfrak{P}_m}.$$

(5) $$\overline{K}_{pm} . \equiv . (p, m) \ni p^{B_0 \mathfrak{P}_{-m}}.$$

(6)§ $$K_{pm_1m_2} . \equiv . (p, m_1, m_2) \ni (\exists p_1 \ni \overline{K}_{p_1m_1} . K_{pp_1m_2}).$$

(7) $$\mathfrak{P}_{m_1m_2} . \equiv . [p_{m_1m_2}] . \equiv . [\text{all } p \ni K_{pm_1m_2}].$$

$\mathfrak{P}_m$, $\mathfrak{P}_{-m}$, $\mathfrak{P}_{m_1m_2}$ may be considered as reductions‖ of $\mathfrak{P}$. We speak of the corresponding reduced developments¶ Δ_{R_m}, $\Delta_{R_{-m}}$, $\Delta_{R_{m_1m_2}}$ and of the reduced functions** μ_{R_m}, $\mu_{R_{-m}}$, $\mu_{R_{m_1m_2}}$ and classes** of reduced functions $\mathfrak{M}_{R_m}$, $\mathfrak{M}_{R_{-m}}$, $\mathfrak{M}_{R_{m_1m_2}}$.

(8)†† $$\begin{aligned} \mathfrak{M}^{B_3} . &\equiv . \mathfrak{M} \ni (\exists \mu \ni m . \supset . \mu^{B_2(\mathfrak{P}_{-m})}). \\ &\equiv . \mathfrak{M} \ni \big(\exists \mu \ni (m . \supset . \exists e_m \ni \overline{K}_{pm} . \supset . \mu_p \geqq e_m)\big). \end{aligned}$$

* $\mathfrak{M}$ is said to have the property B_2 in case $\mathfrak{M}$ is such that there exists a function μ of $\mathfrak{M}$, uniformly bounded from 0.

† (2) is a special case of (1) where the class $\mathfrak{M}$ consists of a single function μ.

‡ The class $\mathfrak{P}_{-m}$ is the class of all elements each developed at every stage $m_0 \geqq m$.

§ p, m_1, m_2 are in the relation $K_{pm_1m_2}$ in case there exists an element p_1 developed at every stage $m_0 \geqq m_1$, $(\overline{K}_{p_1m_1})$, such that p is directly connected with p_1 at some stage $m' \geqq m_2$, $(K_{pp_1m_2})$.

‖ § 11.

¶ § 22.

** § 11.

†† $\mathfrak{M}$ has the property B_3 in case there exists a function μ of $\mathfrak{M}$ such that for every m, μ is uniformly bounded from 0 on $\mathfrak{P}_{-m}$.

39*a*. *Propositions.*—

(1) $$m_1 \geqq m_2 :\supset : \mathfrak{P}_{-m_2}{}^{B_0 \mathfrak{P}_{m_1}} . \mathfrak{P}_{-m_2}{}^{B_0 \mathfrak{P}_{-m_1}} . \overline{K}_{pm_2} . \supset . \overline{K}_{pm_1}.$$

(2)* $$\Delta . \mathfrak{M}^{K_1} :\supset : (\exists m \ni \exists \mathfrak{P}_{-m}) \smile (\mu . p . \supset . \mu_p = 0).$$

(3) $$\Delta . m . \supset . \Delta^{C_1(\mathfrak{P}_{-m})}.$$

(4)† $$\Delta . \mathfrak{M}^{K_2} . m . \supset . \mathfrak{M}^{B_1 \mathfrak{M}^{L}_{-m'}}.$$

(5) $$\mathfrak{M}^{B_3} . \supset . \mathfrak{M}_L{}^{B_3} . \mathfrak{M}_*{}^{B_3}.$$

(6)‡ $$\mathfrak{M}'^{B_3'} . \mathfrak{M}''^{B_3''} . \supset . (\mathfrak{M}'\mathfrak{M}'')^{B_3} . (\mathfrak{M}'\mathfrak{M}'')_L{}^{B_3} . (\mathfrak{M}'\mathfrak{M}'')_*{}^{B_3}.$$

(7)§ $$\Delta . \mathfrak{M}^{LK_1K_2B_3} . m . \supset . \mathfrak{M}_{R_{-m}}{}^{LK_1K_2B_2 . B_1 \mathfrak{P} . D}.$$

(8) $$m_1 . m_3 > m_2 :\supset : \mathfrak{P}_{m_1 m_3}{}^{B_0 \mathfrak{P}_{m_1 m_2}} . \mathfrak{P}_{m_2 m_1}{}^{B_0 \mathfrak{P}_{m_3 m_1}} . K_{pm_1m_3} . \supset . K_{pm_1m_2} \; K_{pm_3m_1} . \supset . K_{pm_2m_1}.$$

(9) $$m_1 . m_2 . \mu^{B_2(\mathfrak{P}_{m_1m_2})} . m_3 > m_2 . \supset . \mu^{B_2 \mathfrak{P}_{m_1m_3}}.$$

(10) $$\mathfrak{P} . \Delta . \mathfrak{M}^{LK_2B_3} :\supset : m_1 . \supset . \exists (\mu_1 . m_1') \ni \mu_1^{B_2(\mathfrak{P}_{m_1m_1'})}.$$

Proof.—

(*a*) $$\mu^{K_2\mathfrak{M}} . \equiv . \mu \ni (\exists \mu_0 \ni e :\supset : \exists m_e \ni K_{p_1p_2m_e} . \supset . A(\mu_{p_1} - \mu_{p_2}) \leqq eA\mu_{0p_1}).$$

By (4) and § 39.8 and $\mathfrak{M}^L$ there are available a μ and a μ_0 such that

(*b*) $$\overline{K}_{pm_1} . \supset . \mu_p \geqq 1 . \mu_{0p} \leqq 1;$$

(*c*) $$\overline{K}_{pm_1} . K_{pp_1m_{\frac{1}{2}}} . \supset . A\mu_p \geqq \tfrac{1}{2}.$$

μ and $m_{\frac{1}{2}}$ are available as the μ_1 and m_1' whose existence is affirmed in (10).

(11) $$\mathfrak{P} . \Delta . \mathfrak{M}^{LK_1K_2B_3} .:\supset :. m_1 :\supset : \exists m_2 \ni m \geqq m_2 . \supset . P_{m_1m_2}{}^{B_0\mathfrak{P}_{-m}}.$$

Proof.—

(*a*) $$\mu^{K_1\mathfrak{M}} . \equiv . \mu \ni (\exists \mu_0 \ni e :\supset : \exists m_e' \ni K_{pm_e'} . \supset . A\mu_p \leqq eA\mu_{0p}).$$

By (10), (4) and $\mathfrak{M}^{LB_3}$ there are available μ, μ_0 and m_3 such that

(*b*) $$K_{pm_1m_3} . \supset . A\mu_p \geqq 1,$$

$$\overline{K}_{pm_1} . \supset . A\mu_{0p} \leqq 1.$$

* That is, either there exists a stage m such that $\mathfrak{P}_{-m}$ is an existent class or $\mathfrak{M}^{K_1}$ consists of a single function and that the 0 function.

† § 34.11.

‡ B_3'; B_3''; B_3 are relative to $\mathfrak{P}'$, Δ'; $\mathfrak{P}''$, Δ''; $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$, $\Delta \equiv \Delta'''$ respectively. Cf. § 21*a*.

§ By § 11*a*4, § 22.1, § 32.5. The K_1, K_2 in the conclusion are of course relative to $\Delta_{R_{-m}}$.

(c) $$K_{pm_1m_3} \,.\, K_{pm_{\frac{1}{2}}'} \,.\supset.\, A\mu_{0p} \geqq 2.$$

(d) $$\mu_0^{K_2 p_1} \,.\equiv.\, \mu_0 \ni (\exists\, \mu_3 \ni e :\supset: \exists\, m_e'' \ni K_{p_1p_2m_e''} \,.\supset.$$
$$A(\mu_{0p_1} - \mu_{0p_2}) \leqq eA\mu_{3p_1}).$$

By (4) there is available a μ_3 such that

(e) $$\bar{K}_{pm_1} \,.\supset.\, \mu_{3p} \leqq 1.$$

(f) $$\bar{K}_{p_1m_1} \,.\, K_{p_1pm_{\frac{1}{2}}''} \,.\supset.\, A\mu_{0p} \leqq \tfrac{3}{2}.$$

$$K_{pm_1m_{\frac{1}{2}}''} \,.\supset.\, A\mu_{0p} \leqq \tfrac{3}{2}.$$

(g) $$m_2 \geqq m_3,\, m_{\frac{1}{2}}',\, m_{\frac{1}{2}}'' \,.\, K_{pm_1m_2} \,.\, K_{pm_2} \,.\supset.\, A\mu_{0p} \geqq 2$$
$$K_{pm_1m_2} \,.\supset.\, A\mu_{0p} \leqq \tfrac{3}{2}.$$

(h) $$m_2 \geqq m_3,\, m_{\frac{1}{2}}',\, m_{\frac{1}{2}}'' \,.\, K_{pm_1m_2} \,.\supset.\, {}^{-}K_{pm_2}.$$

(i) $$m \geqq m_2 \,.\supset.\, \mathfrak{P}_{m_1m_2}^{B_3\mathfrak{C}^{n-1}\cdot B_3\mathfrak{C}_{-m}}.$$

(12) $$\mathfrak{P}' \,.\, \Delta' \,.\, \mathfrak{M}'^{LD_1K_1'B_3'} \,.\, \mathfrak{P}'' \,.\, \mathfrak{M}''^{D} :\supset: \theta^{B_0 m_1' m_1''} \,.\supset.\, \theta^{K_2' m_1' m_1''}.$$

Proof.*—

(a)† $$\mathfrak{M}'^{D_1K_1'} \,.\, \mathfrak{M}''^{D_1} :\supset: \theta^{B_1 m_1' m_1''} \,.\supset.\, \theta^{K_1' m_1' m_1''}.$$

(b) $$\theta^{K_1' m_1' m_1''} \,.\equiv.\, \theta \ni (\exists\, \mu_1'\, \mu_1'' \ni e :\supset: \exists\, m_e' \ni K_{p'm_e'}' \,.\supset.$$
$$A\theta_{p'p''} \leqq eA\mu_{1p'}'\mu_{1p''}'' \quad (p'')).$$

(c) $$(\mu_1'\, \mu_1'')^{K_2' m_1' m_1''} \,.\equiv.\, \mu_1'\, \mu_1'' \ni (\exists\, \mu_2'\, \mu_2'' \ni e :\supset: \exists\, m_e'' \ni K_{p_1'p_2'm_e''} \,.\supset.$$
$$A(\mu_{1p_1'}'\, \mu_{1p''}'' - \mu_{1p_2'}'\, \mu_{1p''}'') \leqq eA\mu_{2p_1'}'\, \mu_{2p''}'' \quad (p'')).$$
$$K_{p_1'p_2'm_1''}' \,.\supset.\, A\mu_{1p_2'}'\, \mu_{1p''}'' \leqq A\mu_{1p_1'}'\, \mu_{1p''}'' + A\mu_{2p_1'}'\, \mu_{2p''}'' \quad (p'').$$

(d)‡ $$\exists\, \mu_0'\mu_0'' \ni (e :\supset: \exists\, m_e \ni K_{p_1'm_e}' \,.\, K_{p_2'm_e}' \,.\, K_{p_1'p_2'm_e}' \,.\supset.$$
$$A\theta_{p_1'p''} + A\theta_{p_2'p''} \leqq eA\mu_{0p_1'}'\, \mu_{0p''}'' \quad (p'')).$$

(e)§ $$\mathfrak{M}'^{DK_2'} \,.\, \mathfrak{M}''^{D} :\supset: \theta^{B_1 m_1' m_1''} \,.\supset.\, \theta^{K_2' m_1' m_1''}.$$

(f)‖ $$m \,.\supset.\, \theta_{R_{-m}}^{K_2' m_1' R_{-m} m_1''}.$$

* In case the development Δ' of $\mathfrak{P}'$ is such that for every m the class $\mathfrak{P}_{-m}'$ is the null class, (12) is at once seen to hold in view of (2).

† § 29II.

‡ By (b), (c) and $\mathfrak{M}'^{D_1}$.

§ I. G. A., § 73.1$_2$, 5$_2$.

‖ By (7) and (12e) . K_2' is relative to $\Delta_{R_{-m}}'$.

$$(g)\quad m\,.\supset.\,\exists\,\mu'_m\mu''_m\ni\big(e:\supset:\exists\,m_{em}\ni\overline{K}'_{p_1'm}\,.\,\overline{K}'_{p_2'm}\,.\,K'_{p_1'p_2'm_{em}}.\supset.$$
$$A(\theta_{p_1'p''}-\theta_{p_2'p''})\leqq eA\mu'_{mp_1'}\mu''_{mp''}\ (p'')\big).$$

Since $\mathfrak{M}''$ has the property D there exists a single function μ'' available as μ''_0 in (d) and as μ''_m in each of the denumerable infinitude of instances in (g). Also on account of § 39a4 and $\mathfrak{M}'^{D_1B_3'}$ there exists a single function μ' available as μ'_0 in (d) and as μ_m' in each of the instances in (g).

$$(h)\qquad\exists\,\mu'\mu''\ni$$

$$(h')\quad e:\supset:\exists\,m'_e\ni K'_{p_1'm_e'}\,.\,K'_{p_2'm_e'}\,.\,K'_{p_1'p_2'm_e'}\,.\supset.$$
$$A\theta_{p_1'p''}+A\theta_{p_2'p''}\leqq eA\mu'_{p_1'}\mu''_{p''}\ (p'');$$

$$(h'')\quad m\,.\,e:\supset:\exists\,m_{e,m}\ni\overline{K}'_{p_1'm}\,.\,\overline{K}'_{p_2'm}\,.\,K'_{p_1'p_2'm_{e,m}}.\supset.$$
$$A(\theta_{p_1'p''}-\theta_{p_2'p''})\leqq eA\mu'_{p_1'}\mu''_{p''}\ (p'').$$

$$(i)^*\qquad m'_e:\supset:\exists\,m''_e>m'_e\ni m\geqq m''_e\,.\supset.\,\mathfrak{P}'^{\,B_0\mathfrak{P}'-m}_{m_e'm}.$$

$$(j)\quad m''_e:\supset:\exists\,m'''_e>m''_e\ni\overline{K}'_{p_1'm_e''}\,.\,\overline{K}'_{p_2'm_e''}\,.\,K'_{p_1'p_2'm_e'''}$$
$$.\supset.\,A(\theta_{p_1'p''}-\theta_{p_2'p''})\leqq eA\mu'_{p_1'}\mu''_{p''}\ (p'').$$

$$(k)\quad p_1'\,.\,p_2'\,.\,m'_e\,.\supset.\,(K'_{p_1'm_e'}\,.\,K'_{p_2'm_e'})\smile(K'_{p_1'm_e'}\,.\,\overline{K}'_{p_2'm_e'})$$
$$\smile(K'_{p_2'm_e'}\,.\,\overline{K}'_{p_1'm_e'})\smile(\overline{K}'_{p_1'm_e'}\,.\,\overline{K}'_{p_2'm_e'}).$$

$$(l)^\dagger\quad p_1'\,.\,p_2'\,.\,K'_{p_1'p_2'm_e''}:\supset:\overline{K}'_{p_1'm_e'}\,.\supset.\,\overline{K}'_{p_1'm_e''}\,.\,\overline{K}'_{p_2'm_e''}\,.$$
$$\overline{K}'_{p_2'm_e'}\,.\supset.\,\overline{K}'_{p_2'm_e''}\,.\,\overline{K}'_{p_1'm_e''}.$$

$$(m)\quad p_1'\,.\,p_2'\,.\,K'_{p_1'p_2'm_e''}.\supset.\,(K'_{p_1'm_e'}\,.\,K'_{p_2'm_e'})\smile(\overline{K}'_{p_1'm_e''}\,.\,\overline{K}'_{p_2'm_e''}).$$

$$(n)\qquad K'_{p_1'p_2'm_e'''}\,.\supset.\,A(\theta_{p_1'p''}-\theta_{p_2'p''})\leqq eA\mu'_{p_1'}\mu''_{p''}\ (p'').$$

$$(o)\quad\exists\,\mu'\mu''\ni(e\,.\supset.\,\exists\,m_e\ni K'_{p_1'p_2'm_e}\,.\supset.\,A(\theta_{p_1'p''}-\theta_{p_2'p''})\leqq eA\mu'_{p_1'}\mu''_{p''}\ (p'')\big).$$

$$(13)^\ddagger\qquad\mathfrak{P}'\,.\,\Delta'\,.\,\mathfrak{M}'^{D_1K_{12}'B_3'}\,.\,\mathfrak{P}''\,.\,\mathfrak{M}''^{D}:\supset:\theta^{B_0\mathfrak{P}'\mathfrak{M}'L'\mathfrak{M}''}\text{`}.\supset.\,\theta^{K_2'\mathfrak{M}'L'\mathfrak{M}''}.$$

$$(14)^\S\qquad\mathfrak{P}'\,.\,\Delta'\,.\,\mathfrak{M}'^{D_1K_{12}'B_3'}\,.\,\mathfrak{P}''\,.\,\mathfrak{M}''^{D}:\supset:\theta^{B_0\mathfrak{P}'\mathfrak{M}'\mathfrak{M}''}\text{`}.\supset.\theta^{K_2'\mathfrak{M}'\mathfrak{M}''}.$$

$$(15)\qquad\mathfrak{P}'\,.\,\Delta'\,.\,\mathfrak{M}'^{D_1K_{12}'B_3'}:\supset:\theta'^{B_0\mathfrak{M}'}\text{`}.\supset.\theta'^{K_2'\mathfrak{M}'}.$$

$$(16)\qquad\mathfrak{P}'\,.\,\Delta'\,.\,\mathfrak{M}'^{D_1\Delta'K_{12}'B_3'}\,.\,\mathfrak{P}''\,.\,\Delta''\,.\,\mathfrak{M}''^{D_1\Delta''K_{12}''B_3''}\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1\Delta''K_{12}B_3}$$
$$.\,(\mathfrak{M}'\mathfrak{M}'')_L^{\,D_1\Delta''K_{12}B_3}\,.\,(\mathfrak{M}'\mathfrak{M}'')_*^{\,D_1\Delta''K_{12}B_3}.$$

* By (11) and (8).
† By (i).
‡ By (12); § 9a5; § 23$a4_{12}$; § 39a5.
§ By (13); § 9a5; § 23$a3_2$.
|| By § 12.1; § 21a1, 2, 3, 4; § 39a5, 15; § 29.3.

40. *Theorems.*—By § 39*a*14, 15; § 23*a*16; § 29.6; § 9*a*5; § 18.2, 3; § 29.3; § 39*a*5 the following theorem is readily proved.

THEOREM I. *From*

$$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{D_1 \Delta K_{12} B_3} \qquad (B_3 \text{ defined in § 39.8})$$

it follows that

$$\mathfrak{M}^{K_{12*}} \,.\, (B_0 \mathfrak{M}_* \,.\, \sim \,.\, K_{12*}) \,.\, \mathfrak{M}_L{}^{D_1 \Delta K_{12} B_3 K_{12*}} \,.\, \mathfrak{M}_*{}^{D_1 \Delta K_{12} B_3 K_{12*}}.$$

This theorem is to be compared with I. G. A.,* § 82II where the property K_{12*} is secured as a result of D, Δ, K_{12}, the development being as here unconditioned. The properties Δ, K_1, K_2 are common to the hypotheses of the two theorems. The property D_1 is less restrictive than D, and it is easier to ascertain whether or not it is possessed by a given class of functions. The genus $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M}^{B_3})$ is very extensive and includes the systems I; II_n; III; III_0; III_e; IV with the ordinary developments of the classes $\mathfrak{P}$ concerned. In particular any class of functions containing a single constant function not zero belongs to this genus. Usually one may ascertain by inspection whether or not a class of functions possesses the property B_3. The hypotheses of the theorem given here cover cases not covered by the theorem of I. G. A., § 82II. Consider for example the system $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$ where

$\mathfrak{P} \equiv$ interval $0 < p \leqq 1$;

$\mathfrak{M} \equiv \left[\text{all } \sum_{i=-k}^{i=+l} a_i p^i\right]$ where k and l may be any positive integers or 0;

$\Delta \equiv$ any development of $\mathfrak{P}$ such that $\mathfrak{M}$ has the properties Δ, K_1, K_2. $\mathfrak{M}$ does not possess the property D. It might be thought that a development of $\mathfrak{P}$ could be designated such that $\mathfrak{M}$ would have the properties Δ, K_1, K_2, $^{-}K_{12*}$. However, this class of functions does have the property D_1 and whatever the development of $\mathfrak{P}$, $\mathfrak{M}$ has the property B_3. Therefore if the development of $\mathfrak{P}$ is such that $\mathfrak{M}$ has the properties Δ, K_1, K_2, then $\mathfrak{M}$ has also the property K_{12*}.

THEOREM II. *From*

$$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{LCD_1 \Delta K_{12} B_3}$$

it follows† *that*

$$\mathfrak{M}^{K_{12*}\cdot A} \,.\, (B_0 \mathfrak{M} \sim K_{12} \mathfrak{M}).$$

* Cf. § 23*a*18.

† By Theorem I and § 36I.

THEOREM III. *From*

$$\mathfrak{P}' \,.\, \Delta' \,.\, \mathfrak{M}'^{D_1\Delta' K_{12}' B_3'} \,.\, \mathfrak{P}'' \,.\, \Delta'' \,.\, \mathfrak{M}''^{D_1\Delta'' K_{12}'' B_3''},$$

it follows that the composite systems*:

$$(\mathfrak{A};\ \mathfrak{P}'\mathfrak{P}'';\ \Delta'';\ \mathfrak{M}'\mathfrak{M}''); \qquad (\mathfrak{A};\ \mathfrak{P}'\mathfrak{P}'';\ \Delta'';\ (\mathfrak{M}'\mathfrak{M}'')_L);$$

$$(\mathfrak{A};\ \mathfrak{P}'\mathfrak{P}'';\ \Delta'';\ (\mathfrak{M}'\mathfrak{M}'')_*),$$

are such that

$$(\mathfrak{M}'\mathfrak{M}'')^{D_1\Delta'' K_{12} B_3 K_{12*}}; \qquad (\mathfrak{M}'\mathfrak{M}'')_L{}^{D_1\Delta'' K_{12} B_3 K_{12*}}; \qquad (\mathfrak{M}'\mathfrak{M}'')_*{}^{LD_1\Delta'' K_{12} B_3 K_{12*}}.$$

From preceding theorems and propositions the following closure theorem is readily obtained.

THEOREM IV. *The respective genera of all systems*:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M}^P),$$

where

$$P \equiv B_3; \qquad D_1 K_{12} B_3; \qquad D_1 \Delta K_{12} B_3 (K_{12*}),$$

are closed under the five operations A, B, C_0, C_1, C_2:

A: linear extension of classes $\mathfrak{M}$;

*B: *-extension of classes* $\mathfrak{M}$;

C: simultaneous composition of classes $\mathfrak{P}$ *and developments* Δ *and*

C_0: *multiplication of classes* $\mathfrak{M}$;

C_1: *linear composition of classes* $\mathfrak{M}$;

C_2: **-composition of classes* $\mathfrak{M}$,

and the combinations of these operations.

The respective subgenera of all systems obtained by operations A, B, C_1, C_2 and their combinations are the repective genera of all systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M}^{P_0}),$$

where

$$P_0 \equiv B_3; \qquad LD_1 K_{12} B_3; \qquad LD_1 \Delta K_{12} B_3 (K_{12*}).$$

These genera are closed under the operations A, B, C_1, C_2 and their combinations; to these genera belong the systems I; II_n; III; III_0; III_e; IV.

Propositions relative to systems $(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M})$ *where* $\mathfrak{P}$ *is* $\mathfrak{P}^{\mathrm{I}}$; $\mathfrak{P}^{\mathrm{I}}$ *or* $\mathfrak{P}^{\mathrm{II}_2}$; $\mathfrak{P}^{\mathrm{I}}$ *or* $\mathfrak{P}^{\mathrm{II}_2}$ *or* $\mathfrak{P}^{\mathrm{II}_3}$; $\mathfrak{P}^{\mathrm{II}_n}$, §§ 41–44.

41. *Propositions† relative to* $\mathfrak{P}^{\mathrm{I}}$.—

(1)‡ $$\mathfrak{M} \,.\, \supset \,.\, \mathfrak{M}^D.$$

* By Theorem I and § 39*a*16.

† The propositions of §§ 41–44 are proved by a detailed consideration of possible systems satisfying the specified hypotheses.

‡ I. G. A., § 48.9, 10.

(2) $$\mathfrak{M}^{L} . \supset . \mathfrak{M}^{A}.$$

(3) $$\mathfrak{M}^{L} . \supset . \mathfrak{M} = \mathfrak{M}^{\mathrm{I}} \breve{} \mathfrak{M} = [0].$$

(4) $$\Delta . \mathfrak{M}^{\mathrm{I}} . \supset . \mathfrak{M}^{\Delta}.$$

(5)* $$\Delta . \mathfrak{M} = [0] . \supset . \mathfrak{M}^{K_1}.$$

(6) $$\Delta . \mathfrak{M}^{-\Delta^{-}K_1} . \supset . \mathfrak{M}^{-L}.$$

(7) $$\Delta . \mathfrak{M} . \supset . \mathfrak{M}^{K_2}.$$

(8) $$\Delta :\supset : \mathfrak{M}^{K_{1*}} . \sim . \mathfrak{M}^{K_{12*}}.$$

(9) $$\Delta . \mathfrak{M} \neq [0] . \supset . \mathfrak{M}^{K_{2*}}.$$

(10) $$\Delta . \mathfrak{M}^{-L} . \supset . \mathfrak{M}^{K_{2*}}.$$

(11) $$\Delta . \mathfrak{M}^{-K_1} . \supset . \mathfrak{M}^{K_{2*}}.$$

(12) $$\Delta . \mathfrak{M}^{\Delta K_1} . \supset . \mathfrak{M}^{K_{1*}K_{12*}}.$$

(13) $$\Delta . \mathfrak{M}^{K_1} \neq [0] . \supset . \mathfrak{M}^{K_{1*}K_{12*}}.$$

(14) $$\Delta . \mathfrak{M}^{-LK_1} . \supset . \mathfrak{M}^{K_{1*}K_{12*}}.$$

(15) $$\Delta . \mathfrak{M} \neq [0] . \supset . \mathfrak{M}^{K_1} . \sim . \mathfrak{M}^{K_{1*}} . \sim . \mathfrak{M}^{K_{12*}}.$$

42.† *Propositions relative to* $\mathfrak{P}^{\mathrm{II}_2}$ or $\mathfrak{P}^{\mathrm{I}}$.

(1) $$\Delta . \mathfrak{M}^{K_2 - K_{1*} - K_{2*}} \neq [0] . \supset . \mathfrak{M}^{-K_{12*}}.$$

(2) $$\Delta . \mathfrak{M}^{-LK_2 - K_{1*} - K_{2*}} . \supset . \mathfrak{M}^{-K_{12*}}.$$

(3) $$\Delta . \mathfrak{M}^{\Delta K_2 - K_{1*} - K_{2*}} . \supset . \mathfrak{M}^{-K_{12*}}.$$

* This proposition is of general reference, i. e., it holds for $\mathfrak{P}$ general.

† We add certain suggestions as to the proofs of these propositions. (1) follows from § 27.10, 11 and the proposition:

$$\mathfrak{P}^{\mathrm{II}_2} \equiv 1, 2 . \Delta^{-F_1 - C_1} . \mathfrak{M}^{K_2} :\supset : (\mu . \supset . \mu_1 = \mu_2) . \mathfrak{M}^{K_{2*}}.$$

(2) is a corollary of (1). (3) is a corollary of (1) and § 27.10, 13. (4) and (5) are proved through the propositions:

$$\mathfrak{P}^{\mathrm{II}_2} . \Delta . \mathfrak{M}^{\Delta - K_2} . \supset . \mathfrak{M}_L = \mathfrak{M}^{\mathrm{II}_2};$$

$$\mathfrak{P}^{\mathrm{II}_2} . \Delta . \mathfrak{M}^{\Delta - K_2 K_1} . \supset . \Delta^{C_1};$$

$$\mathfrak{P}^{\mathrm{II}_2} . \Delta^{-F_1} . \mathfrak{M}^{\Delta - K_1} . \supset . \mathfrak{M}_L = \mathfrak{M}^{\mathrm{II}_2};$$

$$\mathfrak{P}^{\mathrm{II}_2} . \Delta . \mathfrak{M}^{\Delta - K_1 K_2} . \supset . \Delta^{F_1}.$$

(6) follows through a detailed consideration of admissible classes $\mathfrak{M}^{L}$ and the proposition:

$$\mathfrak{P}^{\mathrm{II}_2} . \mathfrak{M} = \mathfrak{A}_0 (1, k) . k \neq 1, 0 . \mathfrak{M}^{\Delta} . \supset . \Delta^{F_1},$$

where $\mathfrak{A}_0$ is a subclass of $\mathfrak{A}$. (7) is a corollary of (6) and § 20*a*3. (8_i) follows through the propositions:

$$\mathfrak{P}^{\mathrm{II}_2} . \Delta . \mathfrak{M}^{-DK_1} . \supset . \Delta^{C_1};$$

$$\mathfrak{P}^{\mathrm{II}_2} . \Delta . \mathfrak{M}^{-DK_2} . \supset . \Delta^{F_1}.$$

(4) $\Delta \,.\, \mathfrak{M}^{\Delta K_1{}^{-}K_2} \,.\supset.\, \mathfrak{M}^{K_{1*}}.$

(5) $\Delta^{-V_1} \,.\, \mathfrak{M}^{\Delta^{-}K_1K_2} \,.\supset.\, \mathfrak{M}^{K_{2*}}.$

(6) $\Delta^{-V_1} \,.\, \mathfrak{M}^{L\Delta} \,.\supset.\, \mathfrak{M}^{A}.$

(7) $\Delta \,.\, \mathfrak{M}^{L^{-}A\Delta} \,.\supset.\, \mathfrak{M}^{K_2}.$

(8_i) $\Delta \,.\, \mathfrak{M}^{-DK_i} \,.\supset.\, \mathfrak{M}^{K_{i*}} \qquad (i = 1, 2, 12).$

43. *Propositions relative to* $\mathfrak{P}^{\mathrm{II}_3}$ *or* $\mathfrak{P}^{\mathrm{II}_2}$ *or* $\mathfrak{P}^{\mathrm{I}}$.

(1)* $\Delta \,.\, \mathfrak{M}^{-D\Delta K_1K_2} \,.\supset.\, \mathfrak{M}^{K_{1*}} \smile \mathfrak{M}^{K_{2*}}.$

(2)* $\Delta \,.\, \mathfrak{M}^{-DK_1K_2K_{12*}} \,.\supset.\, \mathfrak{M}^{K_{1*}} \smile \mathfrak{M}^{K_{2*}}.$

44. *Propositions relative to* $\mathfrak{P}^{\mathrm{II}_n}$.

(1)† $\mathfrak{M}^{L} \,.\supset.\, \mathfrak{M}^{CD}.$

(2)‡ $\Delta^{(FC)_2} \,.\, \mathfrak{M}^{\Delta} \,.\supset.\, \mathfrak{M}_L = \mathfrak{M}^{\mathrm{II}_n}.$

(3)§ $\Delta \,.\, \mathfrak{M}^{K_2K_{1*}} \,.\supset.\, \mathfrak{M}^{K_{12*}}.$

(4)§ $\Delta \,.\, \mathfrak{M}^{K_1K_{2*}} \,.\supset.\, \mathfrak{M}^{K_{12*}}.$

(5) $\Delta' \,.\, \mathfrak{M}'^{\Delta'K_1'K_2'} \,.\supset.\, \mathfrak{M}'^{K_{12}'*}.$

To prove (5) consider $\mathfrak{P}'$ as separated into three mutually exclusive classes $\mathfrak{P}_r'$, $\mathfrak{P}_s'$, $\mathfrak{P}_t'$ defined as follows:

$$\mathfrak{P}_t' \equiv [\text{all } p_t' \ni \mu' \,.\supset.\, \mu'_{p_t'} = 0];$$

$$\mathfrak{P}_s' \equiv [\text{all } p_s' \ni (\exists\, \mu' \ni \mu'_{p_s'} \neq 0 \,.\, \exists\, p' \neq p_s' \ni \mu' \,.\supset.\, \mu'_{p'} = \mu'_{p_s'})];$$

$$\mathfrak{P}_r' \equiv [\text{all } p_r' \ni (\exists\, \mu' \ni \mu'_{p_r'} \neq 0 \,.\, p' \neq p_r' \,.\supset.\, \exists\, \mu' \ni \mu'_{p'} \neq \mu'_{p_r'})].\|$$

* These propositions follow from a consideration of the various classes $\mathfrak{M}^{-D}$ on $\mathfrak{P}^{\mathrm{II}_3}$ and then a discussion in each case of the nature of the development Δ required in order that the class $\mathfrak{M}$ may have the other properties required in the hypothesis.

† I. G. A., § 48.7.

‡ From a consideration of the developmental system we show that n functions $\theta_1 \ldots \theta_n$ belong to $\mathfrak{M}_L$ such that

$$\begin{vmatrix} \theta_{11} & \ldots & \theta_{1n} \\ \vdots & & \vdots \\ \theta_{n1} & \ldots & \theta_{nn} \end{vmatrix} \neq 0.$$

§ $\mathfrak{P}^{\mathrm{II}_n}$ may be divided relative to the development into three subclasses $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ such that $\Delta^{C_1(\mathfrak{P}_1)\,C_1(\mathfrak{P}_2)\,C_2(\mathfrak{P}_3)}$. Consider the further nature of the development of $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ and of the functions belonging to $\mathfrak{M}$ as required by conditions on $\mathfrak{M}$.

‖ We may think of this classification in another way. $\mathfrak{M}'$ is a class of functions on $\mathfrak{P}'$ to $\mathfrak{A}$ but $\mathfrak{P}'$ may as well be considered as a class of functions on $\mathfrak{M}'$ to $\mathfrak{A}$. In the latter sense $\mathfrak{P}_t'$ is the subclass of $\mathfrak{P}'$ consisting of all zero functions on $\mathfrak{M}'$ to $\mathfrak{A}$. $\mathfrak{P}_s'$ is the class of all repeated non-zero functions belonging to $\mathfrak{P}'$ and $\mathfrak{P}_t'$ is the class of all unrepeated non-zero functions belonging to $\mathfrak{P}$.

Further separate $\mathfrak{P}'_s$ into classes $\mathfrak{P}'_{s_i}$, $(i = 1 \cdots q)$, such that:

$$(p'_1 \,.\, p'_2)^{B_0 \mathfrak{P}_{s_i}'} : \supset : \mu' \,.\, \supset \,.\, \mu'_{p_1'} = \mu'_{p_2'};$$

$$p_1'^{B_0 \mathfrak{P}_{s_i}'} \,.\, p_2^{B_0 \mathfrak{P}_{s_j}'} \,.\, s_i \neq s_j \,.\, \supset \,.\, \exists\ \mu' \ni \mu'_{p_1'} \neq \mu'_{p_2'}.$$

$$\mathfrak{M}'^{K_1' K_2' \Delta'} \,.\, \supset \,.\, \Delta'^{C_{(\Gamma' \mathfrak{P}_r' + \mathfrak{P}_s' \overline{\Gamma}' \mathfrak{P}_t') F_{\Gamma' \mathfrak{P}_r'}}}.$$

Also since $\mathfrak{M}'^{K_2'}$ the development must be such that any two elements $p'_r p'_s$ or $p'_r p'_t$ or $p'_s p'_t$ or $p'_s p'_{s_j}$ are ultimately separated by the development.

$$\mathfrak{M}'^{K_2' \Delta'} \,.\, \mathfrak{P}'_{s_i} \,.\, \supset \,.\, \exists\, m_0 \ni (m \geqq m_0 \,.\, \supset \,.\, \exists\, l^m \ni \mathfrak{P}'_{s_i} = \mathfrak{P}'^m).$$

Having thus characterized the development it is not difficult by a consideration of the developmental system and a determinantal consideration as in (2) to show that

$$\mathfrak{M}'_L = [\text{all } \mu' \ni \mu' = \text{constant } (\mathfrak{P}'_{s_i})\ (i) \,.\, \mu' = 0\ (\mathfrak{P}'_t)].$$

$$\mathfrak{M}' \,.\, \mathfrak{M}''^{LCD} \,.\, \supset \,.\, (\mathfrak{M}'\mathfrak{M}'')_L = (\mathfrak{M}'\mathfrak{M}'')_*$$

$$= [\text{all } \varphi^{B_0 \mathfrak{M}'' p'} \ni \varphi_{p''} = \text{constant } (\mathfrak{P}'_{s_i})\ (ip'') \,.\, \varphi_{p''} = 0\ (\mathfrak{P}'_t)\ (p'')].$$

$$\mathfrak{F}_1 \equiv [\text{all } \varphi^{K_1' \mathfrak{M}' \mathfrak{M}'' \,.\, B_0 \mathfrak{M}'' (p')}] = [\text{all } \varphi^{B_0 \mathfrak{M}'' p'} \ni \varphi_{p''} = 0\ (\mathfrak{P}'_t)\ (p'')].$$

$$\mathfrak{F}_2 \equiv [\text{all } \varphi^{K_2' \mathfrak{M}' \mathfrak{M}'' \,.\, B_0 \mathfrak{M}'' p''}] = [\text{all } \varphi^{B_0 \mathfrak{M}'' (p')} \ni \varphi_{p''} = \text{constant } (\mathfrak{P}'_{s_i})\ (ip'')].$$

$$\mathfrak{F}_{12} \equiv [\text{all } \varphi^{K_{12}' \mathfrak{M}' \mathfrak{M}'' \,.\, B_0 \mathfrak{M}'' p''}] = \cap\ (\mathfrak{F}_1 \mathfrak{F}_2) = (\mathfrak{M}'\mathfrak{M}'')_*.$$

The complete existential theory of the eight properties:

$$L;\ C;\ D;\ A;\ \Delta;\ K_1;\ K_2;\ K_{12*}. \quad §§ 45\text{–}48.$$

45. *The complete existential theory of the properties* $L; C; D; A$.— Moore shows* that the properties $L; C; D; A$ are completely independent.† Of the $2^4 = 16$ composite properties formed from $L; C; D; A$ and their negatives it is shown that only five, viz.,

$$(+\ +\ +\ +);\quad (-\ \cdot\ +\ \cdot),‡$$

are possessed by classes of functions on $\mathfrak{P}$ singular and only ten, viz.,

$$(+\ +\ +\ \cdot);\quad (-\ \cdot\ \cdot\ \cdot)$$

are possessed by classes of functions on $\mathfrak{P}$ finite.

* I. G. A., § 48.

† See I. G. A., § 47 and preface to this paper for definition of complete independence.

‡ The dot · indicates that the corresponding composite property may be taken either positively or negatively.

The ten composite properties

$$(+\ +\ +\ \cdot);\quad (-\ \cdot\ \cdot\ \cdot),$$

are all possessed by classes of functions on $\mathfrak{P}$ dual and the 16 composite properties are all possessed by classes of functions on $\mathfrak{P}$ denumerable.

46. *The complete existential theory of the properties* L; C; D; A; Δ; K_1; K_2; K_{12*}.—Certain general relations among these properties are expressed by the propositions § $23a11_{12}$, 14_{12}, 17, 18.* The effect of these propositions is to cut down the number of existent composite properties from 256 to 148. In fact by § $23a11_{12}$ all existent composite properties are of the form

$$(\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ -)\quad \text{or}\quad (\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ +\ +\ +),$$

and by § $23a14_{12}$, 17 none of these can be of the form:

$$(\cdot\ \cdot\ +\ \cdot\ +\ +\ +\ -)\quad \text{or}\quad (+\ +\ \cdot\ -\ \cdot\ +\ +\ +).$$

The following table classifies the existent composite properties as to L; C; D; A giving the number of existent composite properties for $\mathfrak{P}$ unrestricted, $\mathfrak{P}$ finite, $\mathfrak{P}$ dual, and $\mathfrak{P}$ singular.

$L\ C\ D\ A$	$\mathfrak{P}$	$\mathfrak{P}^{\mathrm{II}_n}$	$\mathfrak{P}^{\mathrm{II}_2}$	$\mathfrak{P}^{\mathrm{I}}$
+ + + +	9	9	9	4
− + + +	9	9	9	4
− + + −	9	9	9	4
− − + +	9	9	9	4
− − + −	9	9	9	4
+ + + −	7	7	5	0
− + − +	10	9	8	0
− + − −	10	9	8	0
− − − +	10	9	8	0
− − − −	10	9	8	0
+ + − +	10 or 9†	0	0	0
+ − + +	9	0	0	0
+ + − −	8 or 7†	0	0	0
+ − + −	9	0	0	0
+ − − +	10 or 9†	0	0	0
+ − − −	10	0	0	0
Total	145~146~147~148	88	82	20

* § $23a18$ follows from § $23a14_{12}$, 17.

† We have neither theorem nor examples to show that the composite properties $(+\ +\ -\ +\ +\ +\ +\ -)$, $(+\ +\ -\ -\ +\ +\ +\ -)$, $(+\ -\ -\ +\ +\ +\ +\ -)$, are existent or non-existent.

The propositions of § 44.1, 5; § 42.7, 8_{12}; § 41.1, 2, 6, 7, 12, 14 express relations among the properties L; C; D; A; Δ; K_1; K_2; K_{12*} as properties of classes of functions on $\mathfrak{P}$ finite; $\mathfrak{P}$ dual; $\mathfrak{P}$ singular. The effect of these propositions is to cut down the number of existent composite properties to 88 for $\mathfrak{P}$ finite, to 82 for $\mathfrak{P}$ dual, and to 20 for $\mathfrak{P}$ singular.

The reader will have no difficulty in verifying the table by means of the propositions cited above, and the lists of examples given in the sequel. The examples given are all examples of classes of functions on $\mathfrak{P}^{III}$, $\mathfrak{P}^{II_2}$, $\mathfrak{P}^{II_2}$, or $\mathfrak{P}^{I}$ so that of the 145 composite properties proved to be existent there is no loss in passing from $\mathfrak{P}$ unrestricted to $\mathfrak{P}$ denumerable, nor in passing from $\mathfrak{P}^{II_\alpha}$ to $\mathfrak{P}^{II_3}$. The 88 existent composite properties on $\mathfrak{P}^{II_3}$ include all existent non-linear composite properties.

THEOREM I. *Of the* $2^8 = 256$ *propositions involved in the complete existential theory of the properties*:

$$L;\ C;\ D;\ A;\ \Delta;\ K_1;\ K_2;\ K_{12*},$$

of systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M}),$$

145 *are propositions of existence and* 108 *are propositions of non-existence.* Of the properties*:

$$L;\ C;\ D;\ A;\ \Delta;\ K_1;\ K_2,$$

any six are completely independent. In addition the properties in each of the sets of five:

$$(C;\ D;\ A;\ \Delta;\ K_{12*}),\ (L;\ D;\ A;\ \Delta;\ K_{12*}),\ (L;\ C;\ D;\ \Delta;\ K_{12*}),$$

are completely independent. Any other subset of properties completely independent is a subset of one of the sets already declared to be completely independent.

The proof of the complete independence in the various cases above consists in verifying that the appropriate examples are given in the lists of the sequel. The proof of the last statement of the theorem consists in showing by means of the propositions referred to above that no subset of seven properties is completely independent, that no other subset of six or five properties is completely independent, and

* The three composite properties about which we have not decided are $(L\,C^-D\,A\,\Delta\,K_1K_2^-K_{12*})$; $(L\,C^-D^-A\,\Delta\,K_1K_2^-K_{12*})$; $(L^-C^-D\,A\,\Delta\,K_1K_2^-K_{12*})$.

in verifying that any subset of four, three or two completely independent properties is already included in the subsets of six and five completely independent properties.

47. *Special developments of particular classes* $\mathfrak{P}$.—In order to be able to give the independence examples previously referred to in a concise form we indicate here certain special developments. The symbolism here may appear somewhat formidable, but by a little careful attention the reader should find it perfectly clear. The classes of elements $\mathfrak{P}$ considered are $\mathfrak{P}^{\mathrm{I}}$, $\mathfrak{P}^{\mathrm{II}_2}$, $\mathfrak{P}^{\mathrm{II}_3}$, $\mathfrak{P}^{\mathrm{II}_4}$, $\mathfrak{P}^{\mathrm{III}}$, $\mathfrak{P}^{\mathrm{II}_2\mathrm{III}}$, $\mathfrak{P}^{\mathrm{II}_3\mathrm{III}}$ defined as follows:

$$\mathfrak{P}^{\mathrm{I}} \equiv 1. \quad \mathfrak{P}^{\mathrm{II}_2} \equiv 1, 2. \quad \mathfrak{P}^{\mathrm{II}_3} \equiv 1, 2, 3. \quad \mathfrak{P}^{\mathrm{II}_4} \equiv 1, 2, 3, 4.$$

$$\mathfrak{P}^{\mathrm{III}} \equiv 1, 2, 3, 4, \ldots, n, \ldots$$

$$\mathfrak{P}^{\mathrm{II}_2\mathrm{III}} \equiv \mathfrak{Z}^{\mathrm{II}_2}\mathfrak{Z}^{\mathrm{III}} \equiv [ij] \quad (i = 1, 2; j = 1, 2, \ldots, n, \ldots).$$

$$\mathfrak{P}^{\mathrm{II}_3\mathrm{III}} \equiv \mathfrak{Z}^{\mathrm{II}_3}\mathfrak{Z}^{\mathrm{III}} \equiv [ij] \quad (i = 1, 2, 3; j = 1, 2, \ldots, n, \ldots)$$

47*a*. *Developments of* $\mathfrak{P}^{\mathrm{I}}$.—

$\Delta_1^{\mathrm{I}} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^m = 1)$.*
$\Delta_2^{\mathrm{I}} . \equiv . \Delta \ni (m . \supset . l_m = 1 . {}^{-}\exists\, \mathfrak{P}^{m1})$.†
$\Delta_3^{\mathrm{I}} . \equiv . \Delta \ni (m^{\mathrm{odd}} . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1) . (m^{\mathrm{even}} . \supset . l_m = 1 . {}^{-}\exists\, \mathfrak{P}^{m1})$.
$\Delta_4^{\mathrm{I}} . \equiv . \Delta \ni (m . \supset . l_m = 2 . \mathfrak{P}^{m1} = \mathfrak{P}^{m2} = 1)$.

47*b* *Developments of* $\mathfrak{P}^{\mathrm{II}_2}$.—

$\Delta_1^{\mathrm{II}_2} . \equiv . \Delta \ni (m . \supset . l_m = 2 . \mathfrak{P}^{m1} = 1 . \mathfrak{P}^{m2} = 2)$.
$\Delta_2^{\mathrm{II}_2} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1, 2)$.
$\Delta_3^{\mathrm{II}_2} . \equiv . \Delta \ni (m . \supset . l_m = 1 . {}^{-}\exists\, \mathfrak{P}^{m1})$.
$\Delta_4^{\mathrm{II}_2} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1)$.
$\Delta_5^{\mathrm{II}_2} . \equiv . \Delta \ni (m . \supset . l_m = 2 . \mathfrak{P}^{m1} = \mathfrak{P}^{m2} = 1, 2)$.
$\Delta_6^{\mathrm{II}_2} . \equiv . \Delta \ni (m . \supset . l_m = 1) . (m^{\mathrm{odd}} . \supset . \mathfrak{P}^{m1} = 1, 2) . (m^{\mathrm{even}} . \supset . \mathfrak{P}^{m1} = 1)$.
$\Delta_7^{\mathrm{II}_2} . \equiv . \Delta \ni (m^{\mathrm{odd}} . \supset . l_m = 2 . \mathfrak{P}^{m1} = 1 . \mathfrak{P}^{m2} = 2) . (m^{\mathrm{even}} . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1, 2)$.
$\Delta_8^{\mathrm{II}_2} . \equiv . \Delta \ni (m^{\mathrm{odd}} . \supset . l_m = 2 . \mathfrak{P}^{m1} = \mathfrak{P}^{m2} = 1, 2) . (m^{\mathrm{even}} . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1)$.

* For notations $\mathfrak{P}^m$, l_m, etc., see § 13.
† The class ${}^{m1}\mathfrak{P}$ is the null class.

47*c*. *Developments of* $\mathfrak{P}^{\mathrm{II}_3}$.—

$\Delta_1^{\mathrm{II}_3} . \equiv . \Delta \ni (m . \supset . l_m = 3 . \mathfrak{P}^{m1} = 1 . \mathfrak{P}^{m2} = 2 . \mathfrak{P}^{m3} = 3).$
$\Delta_2^{\mathrm{II}_3} . \equiv . \Delta \ni (m . \supset . l_m = 2 . \mathfrak{P}^{m1} = 1 . \mathfrak{P}^{m2} = 2).$
$\Delta_3^{\mathrm{II}_3} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1).$
$\Delta_4^{\mathrm{II}_3} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1, 2).$
$\Delta_5^{\mathrm{II}_3} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1, 2, 3).$
$\Delta_6^{\mathrm{II}_3} . \equiv . \Delta \ni (m . \supset . l_m = 2 . \mathfrak{P}^{m1} = 1 . \mathfrak{P}^{m2} = 2, 3).$
$\Delta_7^{\mathrm{II}_3} . \equiv . \Delta \ni (m . \supset . l_m = 2 . \mathfrak{P}^{m1} = \mathfrak{P}^{m2} = 1, 2, 3).$

47*d*. *Developments of* $\mathfrak{P}^{\mathrm{II}_4}$.—

$\Delta_1^{\mathrm{II}_4} . \equiv . \Delta \ni (m . \supset . l_m = 4 . \mathfrak{P}^{ml} = l \ (l)).$
$\Delta_2^{\mathrm{II}_4} . \equiv . \Delta \ni (m . \supset . l_m = 2 . \mathfrak{P}^{m1} = 1 . \mathfrak{P}^{m2} = 2, 3).$

47*e*. *Developments of* $\mathfrak{P}^{\mathrm{III}}$.—

$\Delta_1^{\mathrm{III}} . \equiv . \Delta \ni (m . \supset . l_m = m . \mathfrak{P}^{ml} = l \ (l)).$
$\Delta_2^{\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (l_m = m) . (l > 2 . \supset . \mathfrak{P}^{ml} = l) . (l \leqq 2 . \supset . \mathfrak{P}^{ml} = 1, 2)\big).$
$\Delta_3^{\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (l_m = m) . (\mathfrak{P}^{mm} = m, m+1, \ldots, m+n, \ldots) . (l < m . \supset . \mathfrak{P}^{ml} = l)\big).$
$\Delta_4^{\mathrm{III}} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1).$
$\Delta_5^{\mathrm{III}} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1, 2, 3, \ldots, n, \ldots).$
$\Delta_6^{\mathrm{III}} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 2, 3, 4, \ldots, n, \ldots).$
$\Delta_7^{\mathrm{III}} . \equiv . \Delta \ni (m . \supset . l_m = m) . (m^{\mathrm{odd}} . l . \supset . \mathfrak{P}^{ml} = l+1) . (m^{\mathrm{even}} . l . \supset . \mathfrak{P}^{ml} = l).$
$\Delta_8^{\mathrm{III}} . \equiv . \Delta \ni (m . \supset . l_m = m).$
$\big(m^{\mathrm{even}} :\supset : (\mathfrak{P}^{m1} = 1, 2) . (\mathfrak{P}^{mm} = m+1, m+2, \ldots, m+n, \ldots) . (l \neq 1, m . \supset . \mathfrak{P}^{ml} = l+1)\big).$
$\big(m^{\mathrm{odd}} :\supset : (\mathfrak{P}^{mm} = m, m+1, \ldots, m+n, \ldots) . (l < m . \supset . \mathfrak{P}^{ml} = l)\big).$
$\Delta_9^{\mathrm{III}} . \equiv . \Delta \ni (m . \supset . l_m = 1 . \mathfrak{P}^{m1} = 1, 2).$
$\Delta_{10}^{\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (l_m = m) . (l < m . \supset . \mathfrak{P}^{ml} = l+1) . (\mathfrak{P}^{mm} = m+1, m+2, \ldots, m+n, \ldots)\big).$
$\Delta_{11}^{\mathrm{III}} . \equiv . \Delta \ni (m^{\mathrm{even}} . \supset . \Delta^m = \Delta_3^{\mathrm{III}m}) . (m^{\mathrm{odd}} . \supset . \Delta^m = \Delta_{10}^{\mathrm{III}m}).$*

* If m is even then stage m of $\Delta_{11}^{\mathrm{III}}$ is the same as stage m of Δ_3^{III} while if m is odd stage m of $\Delta_{11}^{\mathrm{III}}$ is the same as stage m of $\Delta_{10}^{\mathrm{III}}$.

47*f*. *Developments of* $\mathfrak{P}^{\Pi_2\mathrm{III}}$.—Corresponding to the composition of the classes $\mathfrak{P}^{\Pi_2}$, $\mathfrak{P}^{\mathrm{III}}$ we introduce the composite index ij for l so that stage m has $l_m = i_m j_m$ subclasses $\mathfrak{P}^{mij}$, $i = 1, 2, \ldots i_m$; $j = 1, 2, \ldots j_m$.

$\Delta_1^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni (m :\supset : i_m = 2 . j_m = m . \mathfrak{P}^{mij} = ij \ (ij))$.

$\Delta_2^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni (m . \supset . i_m = 1 . j_m = 1 . \mathfrak{P}^{m11} = 11, 21)$.

$\Delta_3^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni (m . \supset . i_m = 1 . j_m = 1 . \mathfrak{P}^{m11} = 11)$.

$\Delta_4^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni (m . \supset . i_m = 1 . j_m = m . \mathfrak{P}^{m1j} = 1j, 2j \ (j))$.

$\Delta_5^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (i_m = 2 . j_m = m) . (j \geqq 2 . i . \supset . \mathfrak{P}^{mij} = ij)$
$. (\mathfrak{P}^{m11} = \mathfrak{P}^{m21} = 11, 21)\big)$.

$\Delta_6^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (i_m = 2 . j_m = m) . (j < m . i . \supset . \mathfrak{P}^{mij} = ij)$
$. (\mathfrak{P}^{m1m} = 1m . \mathfrak{P}^{m2m} = 2m, 2m + 1, \ldots, 2m + n, \ldots)\big)$.*

$\Delta_7^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (i_m = 2 . j_m = m) . (i . j \neq 1, m . \supset . \mathfrak{P}^{mij} = ij)$
$. (\mathfrak{P}^{m11} = \mathfrak{P}^{m21} = 21 . \mathfrak{P}^{m1m} = 1m . \mathfrak{P}^{m2m} = 2m, 2m + 1, \ldots)\big)$.

$\Delta_8^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni (m^{\mathrm{odd}} . \supset . \Delta^m = \Delta_6^{\Pi_2\mathrm{III}_m}) . (m^{\mathrm{even}} . \supset . \Delta^m = \Delta_7^{\Pi_2\mathrm{III}_m})$.

$\Delta_9^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (i_m = 2 . j_m = m) . (j \neq 1, m . i . \supset . \mathfrak{P}^{mij} = ij)$
$. \mathfrak{P}^{m11} = \mathfrak{P}^{m21} = 11, 21 . \mathfrak{P}^{m1m} = 1m . \mathfrak{P}^{m2m}$
$= 2m, 2m + 1, \ldots, 2m + n, \ldots)\big)$.

$\Delta_{10}^{\Pi_2\mathrm{III}} . \equiv . \Delta \ni (m^{\mathrm{odd}} . \supset . \Delta^m = \Delta_1^{\Pi_2\mathrm{III}_m}) . (m^{\mathrm{even}} . \supset . \Delta^m = \Delta_5^{\Pi_2\mathrm{III}_m})$.

47*g*. *Developments of* $\mathfrak{P}^{\Pi_3\mathrm{III}}$.—

$\Delta_1^{\Pi_3\mathrm{III}} . \equiv . \Delta \ni (m . \supset . i_m = 3 . j_m = m . \mathfrak{P}^{mij} = ij \ (ij))$.

$\Delta_2^{\Pi_3\mathrm{III}} . \equiv . \Delta \ni (m . \supset . i_m = 1 . j_m = m . \mathfrak{P}^{m1j} = 1j, 2j, 3j \ (j))$.

$\Delta_3^{\Pi_3\mathrm{III}} . \equiv . \Delta \ni (m . \supset . i_m = 1 . j_m = 1 . \mathfrak{P}^{m11} = 11)$.

$\Delta_4^{\Pi_3\mathrm{III}} . \equiv . \Delta \ni \big(m :\supset : (i_m = 3 . j_m = m) . (j > 1 . i . \supset . \mathfrak{P}^{mij} = ij)$
$. (i . \supset . \mathfrak{P}^{mi1} = 11, 21, 31)\big)$.

$\Delta_5^{\Pi_3\mathrm{III}} . \equiv . \Delta \ni (m . \supset . i_m = 1 . j_m = 1 . \mathfrak{P}^{m11} = 11, 21, 31)$.

$\Delta_6^{\Pi_3\mathrm{III}} . \equiv . \Delta \ni (m^{\mathrm{odd}} . \supset . \Delta^m = \Delta_4^{\Pi_3\mathrm{III}m}) . (m^{\mathrm{even}} . \supset . \Delta^m = \Delta_1^{\Pi_3\mathrm{III}m})$.

48. *Examples.*—In the following sixteen sections, 48*a*–48*p*, we give the 145 independence examples previously mentioned. We classify these examples into sixteen sets. Each set is characterized by one of the sixteen composite properties formed from the properties L; C; D; A and their negatives and with this restriction contains the examples illustrating all known existent composite properties formed from the properties Δ; K_1; K_2; K_{12*} and their negatives. § 48*a* has the caption:

* $2m$, $2m + 1$, etc., are bipartite indices of which the first part is 2 and the second part is m, $m + 1$, etc.

$LCDA$, 9 examples (§ 23a11$_{12}$, 17), 4 on $\mathfrak{P}^{\mathrm{I}}$ (§ 41.6, 7, 12), 9 on $\mathfrak{P}^{\mathrm{II}_2}$. That is, in this set are given nine existent composite properties in each of which the properties L; C; D; A occur positively. The propositions § 23a11$_{12}$, 17 show that there are no other existent composite properties similarly characterized as to L; C; D; A. Four of the examples of the set are on $\mathfrak{P}^{\mathrm{I}}$ and the propositions § 39.6, 7, 12 show that the other composite properties of the set are non-existent for P^{I}. All of the composite properties of the set, nine in number, are existent for $\mathfrak{P}^{\mathrm{II}_2}$. Only five of the examples are given on $\mathfrak{P}^{\mathrm{II}_2}$, but those composite properties existent for $\mathfrak{P}^{\mathrm{I}}$ are also existent for $\mathfrak{P}^{\mathrm{II}_2}$, $\mathfrak{P}^{\mathrm{II}_\infty}$, $\mathfrak{P}^{\mathrm{III}}$, etc. In fact the character of a class $\mathfrak{M}$ as to the properties

$$L;\ C;\ D;\ A;\ \Delta;\ K_1;\ K_2;\ K_{12*},$$

is invariant* under the enlargement of $\mathfrak{P}$ to a class $\mathfrak{P}'$ by the adjunction of a class $\mathfrak{P}''$ having no elements in common with $\mathfrak{P}$, the development of $\mathfrak{P}'$ being identical with the development of $\mathfrak{P}$ and the functions μ of $\mathfrak{M}$ being extended to functions μ' of $\mathfrak{M}'$ vanishing on $\mathfrak{P}''$.

We give each example in the following order:

(*a*) character as to $\Delta\ K_1\ K_2\ K_{12*};\ K_{1*}\ K_{2*}$ in the order indicated;

(*b*) The class $\mathfrak{P}$ of the example;

(*c*) the development Δ of the class $\mathfrak{P}$;

(*d*) The class $\mathfrak{M}$ designated as a class or by the enumeration of the functions belonging to the class;

(*e*) Proofs are given where it seems desirable.

In all the examples in which the property Δ occurs positively, the developmental system satisfies the conditions 1a, 1$'b$ of § 17 and is, therefore, independent of the representative system. The reader will have no difficulty in specifying a developmental system $\mathfrak{D}(\mathfrak{M})$ such that for every m and p

$$\sum_{q} \delta_p^{mq} = 1; \quad \sum_{q} A\delta_p^{mq} = 1; \quad \delta_{p_n}^{mh} = 0 \ \ (h^p).$$

In case the development Δ is vacuous the reader should recall that every class $\mathfrak{M}$ has the property Δ. We do not in general give the developmental systems for the examples, but we do indicate, in footnotes, developmental systems for the examples § 48a8; § 48g1; § 48g10 which seem to be illustrative.

In case the property Δ occurs negatively, to validate the example

* The same phenomenon as to L; C; D; A is noted in I. G. A., § 48.

one needs to show that for no representative system does there exist a developmental system $\mathfrak{D}(\mathfrak{M})$, belonging to the class $\mathfrak{M}$.

If a particular class $\mathfrak{M}'$ has the property K'_{i*} $(i = 1, 2, 12)$ the proof of the same may be made either by recognizing that other properties possessed by $\mathfrak{M}'$ imply K'_{i*} or, if this is not true, then by actual computation and identification of the classes $(\mathfrak{M}'\mathfrak{M}'')_*$ and $\mathfrak{F}_i \equiv [\text{all } \varphi^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}]$ where $\mathfrak{M}''$ is an arbitrary class possessing the properties L; C; D. The class $\mathfrak{F}_{12}$ is usually best computed as the greatest common subclass of the classes $\mathfrak{F}_1$ and $\mathfrak{F}_2$. If for a particular $\mathfrak{M}''^{LCD}$ the class $(\mathfrak{M}'\mathfrak{M}'')_*$ is not the same as the class $[\text{all } \varphi^{K_1'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}]$ then $\mathfrak{M}'$ does not have the property K'_{i*}. The absence of K'_{i*} in the examples given is best shown by taking $\mathfrak{M}''$ as $\mathfrak{A}$ and showing that $\mathfrak{M}'_*$ and $[\text{all } \varphi'^{K_i'\mathfrak{M}'}]$ are not the same.

48*a*. *L C D A*, 9 *examples* (§ 23*a*11$_{12}$, 17), 4 *on* $\mathfrak{P}^{\mathrm{I}}$ (§ 41.6, 7, 12), 9 *on* $\mathfrak{P}^{\mathrm{II}_2}$.—

	Δ	K_1	K_2	K_{12*}	K_{1*}	K_{2*}			
(1)	+	+	+	+	+	−	$\mathfrak{P}^{\mathrm{I}}$	Δ_2^{I}	$\mathfrak{M} \equiv 0.$*
(2)	−	+	+	+	+	−	$\mathfrak{P}^{\mathrm{I}}$	Δ_3^{I}	$\mathfrak{M} \equiv 0.$
(3)	−	+	+	−	−	−	$\mathfrak{P}^{\mathrm{I}}$	Δ_1^{I}	$\mathfrak{M} \equiv 0.$
(4)	+	−	+	−	−	+	$\mathfrak{P}^{\mathrm{I}}$	Δ_2^{I}	$\mathfrak{M} \equiv [a] \equiv \mathfrak{M}^{\mathrm{I}} \equiv \mathfrak{A}.$
(5)	−	−	+	−	−	−	$\mathfrak{P}^{\mathrm{II}_2}$	$\Delta_4^{\mathrm{II}_2}$	$\mathfrak{M} \equiv [0, a] \equiv [\text{all } \mu \ni \mu_1 = 0].$
(6)	−	+	−	−	−	−	$\mathfrak{P}^{\mathrm{II}_2}$	$\Delta_2^{\mathrm{II}_2}$	$\mathfrak{M} \equiv [0, a].$
(7)	+	+	−	−	+	−	$\mathfrak{P}^{\mathrm{II}_2}$	$\Delta_2^{\mathrm{II}_2}$	$\mathfrak{M} \equiv [a, b] \equiv \mathfrak{M}^{\mathrm{II}_2}.$
(8)	+	−	−	−	−	−	$\mathfrak{P}^{\mathrm{II}_2}$	$\Delta_6^{\mathrm{II}_2}$	$\mathfrak{M} \equiv [a, b]$†.
(9)	−	−	−	−	−	−	$\mathfrak{P}^{\mathrm{II}_2}$	$\Delta_6^{\mathrm{II}_2}$	$\mathfrak{M} \equiv [0, a].$

48*b*. $^{-}$*L C D A*, 9 *examples* (§ 23*a*11$_{12}$, 17), 4 *on* $\mathfrak{P}^{\mathrm{I}}$ (§ 41.7, 14), 9 *on* $\mathfrak{P}^{\mathrm{II}_2}$.—

	Δ	K_1	K_2	K_{12*}	K_{1*}	K_{2*}			
(1)	+	+	+	+	+	+	$\mathfrak{P}^{\mathrm{I}}$	Δ_1^{I}	$\mathfrak{M} \equiv 1.$‡
(2)	−	+	+	+	+	+	$\mathfrak{P}^{\mathrm{I}}$	Δ_1^{I}	$\mathfrak{M} \equiv \frac{1}{2}.$
(3)	+	−	+	−	−	+	$\mathfrak{P}^{\mathrm{I}}$	Δ_2^{I}	$\mathfrak{M} \equiv 1.$
(4)	−	−	+	−	−	+	$\mathfrak{P}^{\mathrm{I}}$	Δ_3^{I}	$\mathfrak{M} \equiv \frac{1}{2}.$
(5)	−	+	+	−	−	−	$\mathfrak{P}^{\mathrm{II}_2}$	$\Delta_1^{\mathrm{II}_2}$	$\mathfrak{M} \equiv (1, 0).$§
(6)	+	+	−	−	+	−	$\mathfrak{P}^{\mathrm{II}_2}$	$\Delta_2^{\mathrm{II}_2}$	$\mathfrak{M} \equiv (1, 0), (1, 1).$

* $\mathfrak{M}$ consists of the zero function.

† For this case a developmental system is:

$$\mathfrak{D}(\mathfrak{M}) \ni (\delta^{m1} = (1, 0)\ (m^{\text{even}})\ .\ \delta^{m1} = (1, 1)\ (m^{\text{odd}})).$$

‡ $\mathfrak{M}$ consists of a single function whose functional value is 1.

§ $\mathfrak{M}$ consists of a single function μ such that $\mu_1 = 1$, $\mu_2 = 0$.

(7) $+ - - -; - - : \mathfrak{P}^{II_2}; \Delta_6^{II_2}; \mathfrak{M} \equiv (1, 0), (1, 1).$
(8) $- + - -; - - : \mathfrak{P}^{II_2}; \Delta_2^{II_2}; \mathfrak{M} \equiv (1, 0).$
(9) $- - - -; - - : \mathfrak{P}^{II_2}; \Delta_6^{II_2}; \mathfrak{M} \equiv (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 0).$

48*c*. $^{-}L^{-}C\ D\ A$, 9 *examples* (§ 23*a*11$_{12}$, 17), 4 *on* $\mathfrak{P}^{I}$ (§ 41.7, 14), 9 *on* $\mathfrak{P}^{II_2}$.—

$\Delta\ K_1\ K_2 K_{12*}; K_{1*} K_{2*}$

(1) $+ + + +; + + : \mathfrak{P}^{I}; \Delta_1^{I}; \mathfrak{M} \equiv 1/n\ (n).$*
(2) $- + + +; + + : \mathfrak{P}^{I}; \Delta_1^{I}; \mathfrak{M} \equiv 1/2n\ (n).$
(3) $+ - + -; - + : \mathfrak{P}^{I}; \Delta_2^{I}; \mathfrak{M} \equiv 1/n\ (n).$
(4) $- - + -; - + : \mathfrak{P}^{I}; \Delta_3^{I}; \mathfrak{M} \equiv 1/2n\ (n).$
(5) $- + + -; - - : \mathfrak{P}^{II_2}; \Delta_1^{II_2}; \mathfrak{M} \equiv (1/n, 0)\ (n).$
(6) $+ + - -; + - : \mathfrak{P}^{II_2}; \Delta_2^{II_2}; \mathfrak{M} \equiv (1, 1), (1/n, 0)\ (n).$
(7) $+ - - -; - - : \mathfrak{P}^{II_2}; \Delta_6^{II_2}; \mathfrak{M} \equiv (1, 0), (1/n, 1/n)\ (n).$
(8) $- + - -; - - : \mathfrak{P}^{II_2}; \Delta_2^{II_2}; \mathfrak{M} \equiv (1/n, 0)\ (n).$
(9) $- - - -; - - : \mathfrak{P}^{II_2}; \Delta_6^{II_2}; \mathfrak{M} \equiv (1/2, 0), (1/2n, 1/2n)\ (n).$

48*d*. $^{-}L^{-}C\ D\ ^{-}A$, 9 *examples* (§ 23*a*11$_{12}$, 17), 4 *on* $\mathfrak{P}^{I}$ (§ 41.7, 14), 9 *on* $\mathfrak{P}^{II_2}$.—

$\Delta\ K_1\ K_2 K_{12*}; K_{1*} K_{2*}$

(1) $+ + + +; + + : \mathfrak{P}^{I}; \Delta_1^{I}; \mathfrak{M} \equiv 1, -1/n\ (n).$
(2) $- + + +; + + : \mathfrak{P}^{I}; \Delta_1^{I}; \mathfrak{M} \equiv -1/n\ (n).$
(3) $+ - + -; - + : \mathfrak{P}^{I}; \Delta_2^{I}; \mathfrak{M} \equiv -1/n\ (n).$
(4) $- - + -; - + : \mathfrak{P}^{I}; \Delta_3^{I}; \mathfrak{M} \equiv -1/n\ (n).$
(5) $- + + -; - - : \mathfrak{P}^{II_2}; \Delta_1^{II_2}; \mathfrak{M} \equiv (-1/n, 0)\ (n).$
(6) $+ + - -; + - : \mathfrak{P}^{II_2}; \Delta_2^{II_2}; \mathfrak{M} \equiv (1, 1), (-1/n, 0)\ (n).$
(7) $+ - - -; - - : \mathfrak{P}^{II_2}; \Delta_6^{II_2}; \mathfrak{M} \equiv (1,0), (0,-1), (1/n, 1/n)\ (n).$
(8) $- + - -; - - : \mathfrak{P}^{II_2}; \Delta_2^{II_2}; \mathfrak{M} \equiv (-1/n, 0)\ (n).$
(9) $- - - -; - - : \mathfrak{P}^{II_2}; \Delta_6^{II_2}; \mathfrak{M} \equiv (1/2, 0), (-1/n, -1/n)\ (n).$

48*e*. $^{-}L\ C\ D^{-}A$, 9 *examples* (§ 23*a*11$_{12}$, 17), 4 *on* $\mathfrak{P}^{I}$ (§ 41.7, 14), 9 *on* $\mathfrak{P}^{II_2}$.—

$\Delta\ K_1\ K_2 K_{12*}; K_{1*} K_{2*}$

(1) $+ + + +; + + : \mathfrak{P}^{I}; \Delta_1^{I}; \mathfrak{M} \equiv 1, -\tfrac{1}{2}.$
(2) $- + + +; + + : \mathfrak{P}^{I}; \Delta_1^{I}; \mathfrak{M} \equiv -1.$
(3) $+ - + -; - + : \mathfrak{P}^{I}; \Delta_2^{I}; \mathfrak{M} \equiv -1.$
(4) $- - + -; - + : \mathfrak{P}^{I}; \Delta_3^{I}; \mathfrak{M} \equiv -1.$
(5) $- + + -; - - : \mathfrak{P}^{II_2}; \Delta_1^{II_2}; \mathfrak{M} \equiv (-1, 0).$

* The symbol (n) means for all n.

(6) $+\,+\,-\,-;\ +\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_2^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,1),\ (-1,\,0).$
(7) $+\,-\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_3};\ \Delta_6^{\mathrm{II}_3};\ \mathfrak{M} \equiv (1,\,1),\ (1,\,0),\ (0,\,-1).$
(8) $-\,+\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_2^{\mathrm{II}_2};\ \mathfrak{M} \equiv (-1,\,0).$
(9) $-\,-\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_6^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,-1),\ (-1,\,0).$

48*f*. *L C D*$^{-}$*A*, 7 *examples* (§ 23*a*14$_{12}$, 17), 0 *on* $\mathfrak{P}^{\mathrm{I}}$ (§ 41.2), 5 *on* $\mathfrak{P}^{\mathrm{II}_2}$ (§ 42.7), 7 *on* $\mathfrak{P}^{\mathrm{II}_3}$.—

$\Delta\ K_1\ K_2 K_{12*};\ K_{1*} K_{2*}$

(1) $-\,+\,+\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_1^{\mathrm{II}_2};\ \mathfrak{M} \equiv [a,\,-a] \equiv [\text{all } \mu \ni \mu_1 = -\mu_2].$
(2) $+\,-\,+\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_3^{\mathrm{II}_2};\ \mathfrak{M} \equiv [a,\,-a].$
(3) $-\,-\,+\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_4^{\mathrm{II}_2};\ \mathfrak{M} \equiv [a,\,-a].$
(4) $-\,+\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_2^{\mathrm{II}_2};\ \mathfrak{M} \equiv [a,\,-a].$
(5) $-\,-\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_6^{\mathrm{II}_2};\ \mathfrak{M} \equiv [a,\,-a].$
(6) $+\,+\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_3};\ \Delta_5^{\mathrm{II}_3};\ \mathfrak{M} \equiv [a,\,b,\,\tfrac{1}{2}(a+b)]$
$\equiv [\text{all } \mu \ni \mu_3 = \tfrac{1}{2}(\mu_1 + \mu_2)].$
(7) $+\,-\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_3};\ \Delta_4^{\mathrm{II}_3};\ \mathfrak{M} \equiv [a,\,b,\,a-b]$
$= [\text{all } \mu \ni \mu_3 = (\mu_1 - \mu_2)].$

48*g*. $^{-}$*L C*$^{-}$*D A*, 10 *examples* (§ 23*a*11$_{12}$), 0 *on* $\mathfrak{P}^{\mathrm{I}}$ (§ 41.1), 8 *on* $\mathfrak{P}^{\mathrm{II}_2}$ (§ 42.8$_{12}$), 9 *on* $\mathfrak{P}^{\mathrm{II}_3}$ (§ 44.5), 10 *on* $\mathfrak{P}^{\mathrm{III}}$.

$\Delta\ K_1\ K_2 K_{12*};\ K_{1*} K_{2*}$

(1) $+\,+\,+\,+;\ +\,+\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_1^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,0),\ (0,\,1).$*
(2) $+\,-\,+\,-;\ -\,+\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_4^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,0),\ (0,\,1).$
(3) $-\,-\,+\,-;\ -\,+\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_4^{\mathrm{II}_2};\ \mathfrak{M} \equiv (\tfrac{1}{2},\,0),\ (0,\,\tfrac{1}{2}).$
(4) $+\,+\,-\,-;\ +\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_5^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,0),\ (0,\,1).$
(5) $+\,-\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_8^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,0),\ (0,\,1).$
(6) $-\,+\,-\,-;\ +\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_2^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,0),\ (0,\,1).$
(7) $-\,+\,+\,+;\ +\,+\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_1^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1/2),\ 0),\ (0,\,\tfrac{1}{2}).$
(8) $-\,-\,-\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_2};\ \Delta_6^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1,\,0),\ (0,\,1).$
(9) $-\,+\,+\,-;\ -\,-\ :\ \mathfrak{P}^{\mathrm{II}_3};\ \Delta_1^{\mathrm{II}_3};\ \mathfrak{M} \equiv (1,\,0,\,0),\ (0,\,1,\,0).$
(10) $+\,+\,+\,-;\ -\,+\ :\ \mathfrak{P}^{\mathrm{III}};\ \Delta_1^{\mathrm{III}};$

$$\mathfrak{M} \equiv \big[\text{all } \mu \ni \big(\exists\, p_\mu \ni \mu_{p_\mu} = 1 \,.\, \mu_p = 0\ (p \neq p_\mu)\big)\big]$$
$$+\ [\text{all } \mu \ni (u_1 = 3) \,.\, (p^{\text{odd}} \neq 1 \,.\supset.\, \mu_p = 0)]$$
$$+\ [\text{all } \mu \ni (\mu_1 = 2) \,.\, (p^{\text{even}} \,.\supset.\, \mu_p = 0)].\dagger$$

Proof of $\mathfrak{M}^{C}$.—The three component classes of $\mathfrak{M}$ are closed as

* A developmental system is $\mathfrak{T}(\mathfrak{M}) \ni \big(\delta^{m1} = (1,\,0),\ \delta^{m2} = (0,\,1)\ (m)\big)$.
† A developmental system is

$$\mathfrak{T}(\mathfrak{M}) \ni \big(\delta_{ll}^{ml} = 1\ (p = l) \,.\, \delta_{pp}^{ml} = 0\ (p \neq l)\ (ml)\big).$$

to $\mathfrak{M}^{III}$. Owing to the peculiar initial functional value characterizing the functions of each component class of $\mathfrak{M}$, the closure of each component class as to $\mathfrak{M}^{III}$ insures the closure of $\mathfrak{M}$ as to $\mathfrak{M}^{III}$ and therefore the closure of $\mathfrak{M}$.

Proof of $\mathfrak{M}^{-K_{12*}}$.—

$$\mathfrak{M}_* = \mathfrak{M}^{III}.$$

$$\varphi^{K_{12}\mathfrak{M}} . \supset . \varphi \text{ has infinite number of zeros.}$$

$$\therefore \mathfrak{M}_* \neq [\text{all } \varphi^{K_{12}\mathfrak{M}}].$$

48*h*. $^{-}L^{-}C^{-}D\,A$, 10 *examples* (§ 23*a*11$_{12}$), 0 *on* $\mathfrak{P}^{I}$ (§ 41.1), 8 *on* $\mathfrak{P}^{II_2}$ (§ 42.8$_{12}$), 9 *on* $\mathfrak{P}^{II_3}$ (§ 44.5), 10 *on* $\mathfrak{P}^{III}$.

$\Delta\ K_1\ K_2 K_{12*}; K_{1*} K_{2*}$

(1) $+ + + +; + + : \mathfrak{P}^{II_2}; \Delta_1^{II_2}; \mathfrak{M} \equiv (0, 1/n), (1/n, 0)\ (n).$
(2) $- + + +; + + : \mathfrak{P}^{II_2}; \Delta_1^{II_2}; \mathfrak{M} \equiv (0, 1/2n), (1/2n, 0)\ (n).$
(3) $+ - + -; - + : \mathfrak{P}^{II_2}; \Delta_4^{II_2}; \mathfrak{M} \equiv (1/n, 0), (0, 1/n)\ (n).$
(4) $- - + -; - + : \mathfrak{P}^{II_2}; \Delta_4^{II_2}; \mathfrak{M} \equiv (1/2n, 0), (0, 1/2n)\ (n).$
(5) $+ + - -; + - : \mathfrak{P}^{II_2}; \Delta_3^{II_2}; \mathfrak{M} \equiv (1/n, 0), (0, 1/n)\ (n).$
(6) $+ - - -; - - : \mathfrak{P}^{II_2}; \Delta_8^{II_2}; \mathfrak{M} \equiv (1/n, 0), (0, 1/n)\ (n).$
(7) $- + - -; + - : \mathfrak{P}^{II_2}; \Delta_2^{II_2}; \mathfrak{M} \equiv (1/n, 0), (0, 1/n)\ (n).$
(8) $- - - -; - - : \mathfrak{P}^{II_2}; \Delta_6^{II_2}; \mathfrak{M} \equiv (1/n, 0), (0, 1/n)\ (n).$
(9) $- + + -; - - : \mathfrak{P}^{II_3}; \Delta_1^{II_3}; \mathfrak{M} \equiv (1/n, 0, 0), (0, 1/n, 0)\ (n).$
(10) $+ + + -; - + : \mathfrak{P}^{III}; \Delta_1^{III};$

$$\mathfrak{M} \equiv [\text{all } \mu \ni (\exists p_\mu \ni p^{\text{odd}} \geqq p_\mu . \supset . \mu_p = 0)]$$
$$+ [\text{all } \mu \ni (\exists p_\mu \ni p^{\text{even}} \geqq p_\mu . \supset . \mu_p = 0)].$$

Proof of $\mathfrak{M}^{-C}$.—Consider sequence $\{\mu_n\}$, such that

$$\mu_{np} = 0\ (p^{\text{even}} > n); \quad \mu_{np} = 1\ (p \leqq n); \quad \mu_{np} = 1\ (p^{\text{odd}})$$

and the functions μ_0 and θ such that

$$\mu_{0p} = p\ (p^{\text{even}}); \quad \mu_{0p} = 0\ (p^{\text{odd}}); \quad \theta_p = 1\ (p).$$
$$\lim_n \mu_n = \theta\ (\mathfrak{P}^{III}; \mu_0).$$

For proof of $\mathfrak{M}^{-K_{12*}}$ see § 48*g*10.

48*i*. $^{-}L\,C^{-}D^{-}A$, 10 *examples* (§ 23*a*11$_{12}$), 0 *on* $\mathfrak{P}^{I}$ (§ 41.1), 8 *on* $\mathfrak{P}^{II_2}$ (§ 42.8$_{12}$), 9 on $\mathfrak{P}^{II_3}$ (§ 44.5), 10 *on* $\mathfrak{P}^{III}$.—

$\Delta\ K_1\ K_2 K_{12*}; K_{1*} K_{2*}$

(1) $+ + + +; + + : \mathfrak{P}^{II_2}; \Delta_1^{II_2}; \mathfrak{M} \equiv (0, 1), (1, 0), (-1/2, 0).$
(2) $- + + +; + + : \mathfrak{P}^{II_2}; \Delta_1^{II_2}; \mathfrak{M} \equiv (0, 1/2), (-1/2, 0).$

(3) $+\ -\ +\ -;\ -\ +\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_4^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1, 0),\ (0, -1/2).$
(4) $-\ -\ +\ -;\ -\ +\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_4^{\mathrm{II}_2};\ \mathfrak{M} \equiv (0, 1/2),\ (-1, 0).$
(5) $+\ +\ -\ -;\ +\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_5^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1, 0),\ (0, 1),\ (-1/2, 0).$
(6) $+\ -\ -\ -;\ -\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_8^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1, 0),\ (0, 1),\ (-1/2, 0).$
(7) $-\ +\ -\ -;\ +\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_2^{\mathrm{II}_2};\ \mathfrak{M} \equiv (0, 1),\ (-1/2, 0).$
(8) $-\ -\ -\ -;\ -\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_6^{\mathrm{II}_2};\ \mathfrak{M} \equiv (0, 1),\ (-1/2, 0).$
(9) $-\ +\ +\ -;\ -\ -\ :\mathfrak{P}^{\mathrm{II}_3};\ \Delta_1^{\mathrm{II}_3};\ \mathfrak{M} \equiv (0, 1, 0),\ (-1/2, 0, 0).$
(10) $+\ +\ +\ -;\ -\ +\ :\mathfrak{P}^{\mathrm{III}};\ \Delta_1^{\mathrm{III}};$

$$\mathfrak{M} \equiv \left[\text{all } \mu \ni \left(\exists\, p_\mu \ni \mu_{p_\mu} = 1 \,.\, \mu_p = 0\ (p \neq p_\mu)\right)\right] + [\text{all } \mu \ni (\mu_1 = -3) \,.\, (p^{\text{odd}} \neq 1 \,.\supset.\, \mu_p = 0)] + [\text{all } \mu \ni (\mu_1 = -2) \,.\, (p^{\text{even}} \,.\supset.\, \mu_p = 0)].$$

48*j*. $^{-}L^{-}C^{-}D^{-}A$, 10 *examples* (§ 23*a*11$_{12}$), 0 *on* $\mathfrak{P}^{\mathrm{I}}$ (§ 41.1), 8 *on* $\mathfrak{P}^{\mathrm{II}_2}$ (§ 42.8$_{12}$), 9 on $\mathfrak{P}^{\mathrm{II}_3}$ (§ 44.5), 10 on $\mathfrak{P}^{\mathrm{III}}$.—

$\Delta\ K_1\ K_2\ K_{12*};\ K_{1*}\ K_{2*}$

(1) $+\ +\ +\ +;\ +\ +\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_1^{\mathrm{II}_2};\ \mathfrak{M} \equiv (-2, 0),\ (1/n, 0),\ (0, 1/n)\ \ (n).$
(2) $-\ +\ +\ +;\ +\ +\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_1^{\mathrm{II}_2};\ \mathfrak{M} \equiv (1/n, 0),\ (0, -1)\ \ (n).$
(3) $+\ -\ +\ -;\ -\ +\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_4^{\mathrm{II}_2};\ \mathfrak{M} \equiv (0, -1),\ (1/n, 0)\ \ (n).$
(4) $-\ -\ +\ -;\ -\ +\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_4^{\mathrm{II}_2};\ \mathfrak{M} \equiv (0, -1),\ (1/2n, 0)\ \ (n).$
(5) $+\ +\ -\ -;\ +\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_5^{\mathrm{II}_2};\ \mathfrak{M} \equiv (-2, 0),\ (1/n, 0),\ (0, 1/n)\ \ (n).$
(6) $+\ -\ -\ -;\ -\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_8^{\mathrm{II}_2};\ \mathfrak{M} \equiv (-2, 0),\ (1/n, 0),\ (0, 1/n)\ \ (n).$
(7) $-\ +\ -\ -;\ +\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_2^{\mathrm{II}_2};\ \mathfrak{M} \equiv (0, -1),\ (1/n, 0)\ \ (n).$
(8) $-\ -\ -\ -;\ -\ -\ :\mathfrak{P}^{\mathrm{II}_2};\ \Delta_6^{\mathrm{II}_2};\ \mathfrak{M} \equiv (0, -1),\ (1/n, 0)\ \ (n).$
(9) $-\ +\ +\ -;\ -\ -\ :\mathfrak{P}^{\mathrm{II}_3};\ \Delta_1^{\mathrm{II}_3};\ \mathfrak{M} \equiv (-1/n, 0, 0),\ (0, -1/n, 0)\ \ (n).$
(10)* $+\ +\ +\ -;\ -\ +\ :\mathfrak{P}^{\mathrm{III}};\ \Delta_1^{\mathrm{III}};$

$$\mathfrak{M} \equiv [\text{all } \mu \ni (\mu_1 = -2\breve{}1\breve{}0) \,.\, (\exists\, p_\mu \ni p^{\text{odd}} \geqq p_\mu \,.\supset.\, \mu_p = 0)] + [\text{all } \mu \ni (\mu_1 = -2\breve{}1\breve{}0) \,.\, (\exists\, p_\mu \ni p^{\text{even}} \geqq p_\mu \,.\supset.\, \mu_p = 0)].$$

48*k*. $L\ C^{-}D\ A$, 9 *examples* (§ 23*a*11$_{12}$),† 0 *on* $\mathfrak{P}^{\mathrm{II}_m}$ (§ 44.1), 9 *on* $\mathfrak{P}^{\mathrm{III}}$.—

$\Delta\ K_1\ K_2\ K_{12*};\ K_{1*}\ K_{2*}$

(1) $+\ +\ +\ +;\ +\ -\ :\mathfrak{P}'^{\mathrm{III}};\ \Delta_1'^{\mathrm{III}};$

$$\mathfrak{M}' \equiv [\text{all } \mu'^{\text{ult } 0}] \equiv [\text{all } \mu' \ni (\exists\, p'_{\mu'} \ni p' \geqq p'_{\mu'} \,.\supset.\, \mu'_{p'} = 0)]\ddagger$$

* Cf. § 48*h*10.

† The proposition cited permits 10 examples in this set. But we have no example for ($L\ C^{-}D\ A\ \Delta\ K_1K_2{}^{-}K_{1*}$). Neither have we a proposition precluding the existence of such an example.

‡ Cf. I. G. A., § 48*d*.

Proof of $\mathfrak{M}'^{K_{12}'*}$.—

$\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = \left[\text{all } \phi^{B_0 \mathfrak{M}'', p'')} \ni \left(\exists p'_\phi \ni p' \geqq p'_\phi . \supset . \phi_{p'p''} = 0\ (p'')\right)\right].$
$\mathfrak{F}_1 = (\mathfrak{M}'\mathfrak{M}'')_* . \mathfrak{F}_2 = [\text{all } \phi^{B_0 \mathfrak{M}'')(p')}] . \mathfrak{F}_{12} \equiv \cap\ (\mathfrak{F}_1\mathfrak{F}_2) = \mathfrak{F}_1 = (\mathfrak{M}'\mathfrak{M}'')_*.$

(2) $- + + +; + - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{74}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0} \ni \mu_1 = 0].$
(3) $- + + -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{1}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0} \ni \mu_1 = 0].$
(4) $+ - + -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{4}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0}].$
(5) $- - + -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{4}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0} \ni \mu_1 = 0].$
(6) $+ + - -; + - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{2}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0}].$
(7) $- + - -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{24}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0} \ni \mu_1 = 0].$
(8) $+ - - -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{94}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0}].$
(9) $- - - -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{9}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult } 0} \ni \mu_1 = 0].$

48*l*. *L*$^{-}$*C D A*, 9 *examples* (§ 23*a*11$_{12}$, 17), 0 *on* $\mathfrak{P}^{\text{II}_n}$ (§ 44.1), 9 *on* $\mathfrak{P}^{\text{III}}$.—

$\Delta\ K_1\ K_2 K_{12*}; K_{1*} K_{2*}$
(1) $+ + + +; - +: \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{3};$
$\mathfrak{M} \equiv [\text{all } \mu^{\text{ult. constant}}]^* \equiv [\text{all } \mu \ni (\exists p_\mu \ni p \geqq p_\mu . \supset . \mu_p = \mu_{p_\mu})]$
(2) $- + + -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{3}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}} \ni \mu_1 = 0].$
(3) $- + + +; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{11}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}} \ni \mu_1 = 0].$
(4) $+ - + -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{4}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}}].$
(5) $- - + -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{10}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}} \ni \mu_1 = \mu_2].$
(6) $+ + - -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{5}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}}].$
(7) $- + - -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{5}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}} \ni \mu_1 = 0].$
(8) $+ - - -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{6}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}}].$
(9) $- - - -; - - : \mathfrak{P}^{\text{III}}; \Delta^{\text{III}}_{6}; \mathfrak{M} \equiv [\text{all } \mu^{\text{ult. const.}} \ni \mu_1 = \mu_2].$

48*m*. *L*$^{-}$*C D*$^{-}$*A*, 9 *examples* (§ 23*a*11$_{12}$, 17), 0 *on* $\mathfrak{P}^{\text{II}_n}$ (§ 44.1), 9† *on* $\mathfrak{P}^{\text{III}}$.—

$\Delta\ K_1\ K_2 K_{12*}; K_{1*} K_{2*}$
(1)‡ $+ + + +; + -; \mathfrak{P}^{\text{II}_2\text{III}}; \Delta^{\text{II}_2\text{III}}_{1};$
$\mathfrak{M} \equiv [\text{all } \mu \ni (i . \supset . (\sum_j A\mu_{ij})^{\text{convergent}}) . (\exists j_\mu \ni j \geqq j_\mu . \supset . \mu_{1j} = -\mu_{2j})].$

Proof of $\mathfrak{M}^{-C}$.—Consider sequence $\{\mu_n\}$ such that

$n . \supset . \left(\mu_{n1j} = 1/j^2\ (j) . \mu_{n2j} = 1/j^2\ (j < n) . \mu_{n2j} = -(1/j^2)\ (j > n)\right),$

* Cf. I. G. A., § 48*d*.

† $\mathfrak{P}^{\text{II}_2\text{III}}$ is denumerable.

‡ The following is also an interesting example for this case:

$$\mathfrak{P}^{\text{IV}};\ \Delta^{\text{IV}} \equiv \text{ordinary development of } \mathfrak{P}^{\text{IV}};\ \mathfrak{M} \equiv \left[\text{all } \sum_{i=0}^{n} a_i p^i\right].$$

and functions μ_0 and θ such that

$$\mu_{01j} = -\mu_{02j} = 1/j^2\,(j)\,.\,\theta_{ij} = 1/j^2\,(ij).$$
$$\mathop{\mathbf{L}}_{n}\mu_n = \theta \quad (\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \mu_0).$$

Proof of $\mathfrak{M}^{DK_1}$.—This follows readily from the proposition*
$$\mathfrak{P}^{\mathrm{III}}\mathfrak{M}^{\mathrm{III}_c}\,.\supset.\,\mathfrak{M}^{DK_1}.$$

(2) $-+++;\ --:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_1^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M}_1 \equiv [\mathrm{all}\ \mu^{B_0\mathfrak{M}} \ni \mu_{11} = \mu_{21}].$
(3) $-++\,+;\ --:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_{10}^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M}_1 \equiv [\mathrm{all}\ \mu^{B_0\mathfrak{M}} \ni \mu_{11} = \mu_{21}].$
(4) $+-+-;\ --:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_3^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M} \equiv \mathfrak{M}.$
(5) $--+-;\ --:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_3^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M}_2 \equiv [\mathrm{all}\ \mu^{B_0\mathfrak{M}} \ni \mu_{11} = 0].$
(6) $++--;\ +-:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_5^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M} \equiv \mathfrak{M}.$
(7) $-+--;\ --:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_5^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M}_2 \equiv [\mathrm{all}\ \mu^{B_0\mathfrak{M}} \ni \mu_{11} = 0].$
(8) $+---;\ --:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_2^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M} \equiv \mathfrak{M}.$
(9) $----;\ --:\mathfrak{P}^{\mathrm{II}_2\mathrm{III}};\ \Delta_2^{\mathrm{II}_2\mathrm{III}};\ \mathfrak{M}_2 \equiv [\mathrm{all}\ \mu^{B_0\mathfrak{M}} \ni \mu_{11} = 0].$

48*n*. *L*$^-$*C*$^-$*D A*, 9 *examples* (§ 23*a*11$_{12}$, 17),† 9 *on* $\mathfrak{P}^{\mathrm{II}_n}$ (§ 44.1), 9 *on* $\mathfrak{P}^{\mathrm{III}}$.—

$\Delta\ K_1\ K_2\,K_{12*};\ K_{1*}\,K_{2*}$
(1)‡ $++++;\ --:\mathfrak{P}'^{\mathrm{II}_2\mathrm{III}};\ \Delta'^{\mathrm{II}_2\mathrm{III}}_6;$

$$\mathfrak{M}' \equiv \left[\mathrm{all}\ \mu' \ni \left(\exists\, j_{\mu'} \ni j \geqq j_{\mu'}\,.\supset.\,(\mu'_{1j} = 0\,.\,\mu'_{2j} = \mu'_{2j_{\mu'}})\right)\right]$$
$$\equiv [\mathrm{all}\ \mu' \ni \mu_1'^{\,\mathrm{ult.}\,0}\,.\,\mu_2'^{\,\mathrm{ult.\,const.}}].\S$$

Proof of $\mathfrak{M}'^{K_{12'}}$.—The classes‖ $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_{12}$ here are easily computed. The classes $\mathfrak{M}'$ of the examples of § 48*k*1 and § 48*l*1 each possess the property K'_{12*}. In each of these cases the classes $\mathfrak{F}_{12}$ and therefore the classes $(\mathfrak{M}'\mathfrak{M}'')_*$ are easily computed. From these

* I. G. A., § 23*c*8.

† The reference cited permits ten examples in this set, but we have neither an example for the composite property ($L^-C^-D\,A\,\Delta\,K_1K_2{}^-K_{12*}$) nor a theorem precluding the existence of such an example.

‡ An interesting example (see I. G. A., § 48*d*) for this case is:

$\mathfrak{P} \equiv [ij]\ (i = 1, 2, \ldots, n, \ldots; j = 1, 2, \ldots, n, \ldots);$
$\Delta \ni m\,.\supset.\,(i_m = j_m = m)\,.\,(i < m\,.\,j\,.\supset.\,\mathfrak{P}^{mij} = ij)\,.\,(j\,.\supset.\,\mathfrak{P}^{mmj} = mj,\ m+1\,j,\ \ldots,\ m+n\,j,\ \ldots);$
$\mathfrak{M} \equiv [\mathrm{all}\ \mu \ni (\exists j_\mu \ni j \geqq j_\mu\,.\supset.\,\mu_{ij} = 0(i))\,.\,(j :\supset: \exists\, i_j \ni i \geqq i_j\,.\supset.\,\mu_{ij} = \mu_{i_j j})].$

This example is obtained from the examples of § 48*k*1, § 48*l*1 by the regular composition and linear extension of the classes $\mathfrak{M}$ and the regular composition of classes $\mathfrak{P}$ and developments Δ. The example given above is an example of composition of a different kind.

§§ 48*k*1, § 48*l*1.

‖ Cf. § 48.

considerations it is not difficult to show that in the present case

$$\mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = \mathfrak{F}_{12} = \Big[\text{all } \phi^{B_0 \mathfrak{M}'' \mathfrak{M}' \mathfrak{P}'} \ni \big(\exists\, j_\phi \ni j \geqq j_\phi . \supset . \phi_{1jp''} = 0 \quad (p'')\big)$$
$$. \big(\phi . : \supset : . \exists\, \mu'\mu'' \ni e : \supset : \exists\, j_e \ni j_1 \geqq j_e . j_2 \geqq j_e$$
$$. \supset . A(\phi_{2j_1p''} - \phi_{2j_2p''}) \leqq eA\mu'_{2j_1}\mu''_{1p''} \quad (p'')\big)\Big].$$

(2) $- + + - ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_6 ; \mathfrak{M}'_1 \equiv [\text{all } \mu'^{B_0 \mathfrak{M}'} \ni \mu'_{11} = 0].$

(3) $- + + + ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_4 ; \mathfrak{M}'_1 \equiv [\text{all } \mu'^{B_0 \mathfrak{M}'} \ni \mu'_{11} = 0].$

(4) $+ - + - ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_3 ; \mathfrak{M}' \equiv \mathfrak{M}'.$

(5) $- - + - ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_7 ; \mathfrak{M}'_2 \equiv [\text{all } \mu'^{B_0 \mathfrak{M}'} \ni \mu'_{11} = \mu'_{12}].$

(6) $+ + - - ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_9 ; \mathfrak{M}' \equiv \mathfrak{M}'.$

(7) $- + - - ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_9 ; \mathfrak{M}'_1 \equiv [\text{all } \mu'^{B_0 \mathfrak{M}'} \ni \mu'_{11} = 0].$

(8) $+ - - - ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_2 ; \mathfrak{M}' \equiv \mathfrak{M}'.$

(9) $- - - - ; - - : \mathfrak{P}'^{\mathrm{II}_2\mathrm{III}} ; \Delta'^{\mathrm{II}_2\mathrm{III}}_2 ; \mathfrak{M}'_1 \equiv [\text{all } \mu'^{B_0 \mathfrak{M}'} \ni \mu'_{11} = 0].$

48*o*. *L C⁻D⁻A, 7 examples* (§ 23*a*14$_{12}$)*, 0 *on* $\mathfrak{P}^{\mathrm{II}_n}$ (§ 44.1), 7 *on* $\mathfrak{P}^{\mathrm{III}}$. —We define the following class of functions on $\mathfrak{P}^{\mathrm{II}_2\mathrm{III}}$:

$$\mathfrak{M} \equiv [\text{all } \mu \ni i . \supset . \mu_i^{\text{ult. } 0}].$$

The classes of functions on $\mathfrak{P}^{\mathrm{II}_2\mathrm{III}}$ of this set of examples are all subclasses of $\mathfrak{M}$, the functions of which satisfy certain further specified conditions.

$\Delta\ K_1\ K_2 K_{12*} ; K_{1*} K_{2*}$

(1) $- + + - ; - - : \mathfrak{P}^{\mathrm{II}_2\mathrm{III}} ; \Delta^{\mathrm{II}_2\mathrm{III}}_1 ; \mathfrak{M}_1 \equiv [\text{all } \mu \ni j . \supset . \mu_{1j} = -\mu_{2j}].$

(2) $+ - + - ; - - : \mathfrak{P}^{\mathrm{II}_2\mathrm{III}} ; \Delta^{\mathrm{II}_2\mathrm{III}}_2 ; \mathfrak{M}_2 \equiv [\text{all } \mu \ni (\mu_{11} = \mu_{21})$
$. (j > 1 . \supset . \mu_{1j} = -\mu_{2j})]$

(3) $- - + - ; - - : \mathfrak{P}^{\mathrm{II}_2\mathrm{III}} ; \Delta^{\mathrm{II}_2\mathrm{III}}_3 ; \mathfrak{M}_1 \equiv [\text{all } \mu \ni j . \supset . \mu_{1j} = -\mu_{2j}].$

(4) $+ + - - ; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}} ; \Delta^{\mathrm{II}_3\mathrm{III}}_2 ; \mathfrak{M} \equiv \big[\text{all } \mu \ni \big(i . \supset . \mu_i^{\text{ult. } 0}\big)$
$. \big(j . \supset . \mu_{3j} = 1/2(\mu_{1j} + \mu_{2j})\big)\big].$

(5) $- + - - ; - - : \mathfrak{P}^{\mathrm{II}_2\mathrm{III}} ; \Delta^{\mathrm{II}_2\mathrm{III}}_4 ; \mathfrak{M}_1 \equiv [\text{all } \mu \ni j . \supset . \mu_{1j} = -\mu_{2j}].$

(6) $+ - - - ; - - : \mathfrak{P}^{\mathrm{II}_2\mathrm{III}} ; \Delta^{\mathrm{II}_2\mathrm{III}}_2 ;$
$\mathfrak{M}_3 \equiv [\text{all } \mu \ni (j > 1 . \supset . \mu_{1j} = -\mu_{2j})].$

(7) $- - - - ; - - : \mathfrak{P}^{\mathrm{II}_2\mathrm{III}} ; \Delta^{\mathrm{II}_2\mathrm{III}}_2 ; \mathfrak{M}_1 \equiv [\text{all } \mu \ni j . \supset . \mu_{1j} = -\mu_{2j}].$

* The reference cited permits 8 examples in this set, but we have neither an example for the composite property ($L\,C^-D^-A\,\Delta\,K_1K_2^-K_{12*}$) nor a proposition precluding the existence of such an example.

48*p*. $L^{-}C^{-}D^{-}A$, 10 *examples* (§ 23*a*11$_{12}$), 0 *on* $\mathfrak{P}^{\mathrm{II}_n}$ (§ 44.1), 10 *on* $\mathfrak{P}^{\mathrm{III}}$.—

$\Delta \quad K_1 \quad K_2 K_{12*}; K_{1*} K_{2*}$

(1) $+ + + +; + - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_1^{\mathrm{II}_3\mathrm{III}};$
$\mathfrak{M} \equiv [\text{all } \mu \ni (i \,.\supset.\, (\sum_j A\mu_{ij})^{\text{convergent}}) \,.\, (\mu_1^{\text{ult }0}) \,.$
$(\exists j_\mu \ni j \geqq j_\mu \,.\supset.\, \mu_{2j} = -\mu_{3j})].$*

(2) $- + + -; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_1^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M}_1 \equiv [\text{all } \mu^{B_0\mathfrak{M}} \ni \mu_{11} = \mu_{21} = \mu_{31}].$

(3) $- + + +; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_6^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M}_1 \equiv [\text{all } \mu^{B_0\mathfrak{M}} \ni \mu_{11} = \mu_{21} = \mu_{31}].$

(4) $+ - + -; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_3^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M} \equiv \mathfrak{M}.$

(5) $- - + -; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_3^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M}_2 \equiv [\text{all } \mu^{B_0\mathfrak{M}} \ni \mu_{11} = 0].$

(6) $+ + - -; + - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_4^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M} \equiv \mathfrak{M}.$

(7) $- + - -; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_4^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M}_2 \equiv [\text{all } \mu^{B_0\mathfrak{M}} \ni \mu_{11} = 0].$

(8) $+ - - -; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_5^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M} \equiv \mathfrak{M}.$

(9) $- - - -; - - : \mathfrak{P}^{\mathrm{II}_3\mathrm{III}}; \Delta_5^{\mathrm{II}_3\mathrm{III}}; \mathfrak{M}_2 \equiv [\text{all } \mu^{B_0\mathfrak{M}} \ni \mu_{11} = 0].$

(10) $+ + + -; - - :$

$\mathfrak{P} \equiv [ij] \equiv [ij^i] \,(i = 1, 2, \ldots, n, \ldots; j^i \equiv -i, -i+1, \ldots, 0, 1, 2, \ldots, i);$

Each stage m of the development Δ consists of the classes:

$$\mathfrak{P}^{mij} \equiv ij \,(i = 1, 2, \ldots, m; \qquad j = -i, -i+1, \ldots, 0, 1, 2, \ldots, i);$$

$$\mathfrak{M} \equiv \Big[\text{all } \mu \ni \Big(\exists\, (i_\mu \,.\, \{a_i\}_\mu \,.\, \{b_i\}_\mu) \ni (\mathop{\mathrm{L}}_i a_i = \mathop{\mathrm{L}}_i b_i = 0) \,.$$
$$\Big(i \geqq i_\mu \,.\, j \,.\supset.\, \mu_{ij} = a_i \cos \frac{j\pi}{2i} + b_i \sin \frac{j\pi}{2i}\Big)\Big)\Big].$$

Proof of $\mathfrak{M}^{-C}$.—

$$\exists\, \{\mu_n\}^{B_0\mathfrak{M}} \ni \Big((i \leqq n \,.\, j^i) \,.\supset.\, \mu_{nij} = \frac{1}{i}\Big) \,.\, \Big((i > n \,.\, j^i) \,.\supset.\, \mu_{nij} = \frac{1}{i} \cos \frac{j\pi}{2i}\Big).$$

$$\exists\, \mu_0^{B_0\mathfrak{M}} \ni \Big(\mu_{0ij} = \frac{1}{i^{\frac{1}{2}}} \sin \frac{j\pi}{2i} \quad (ij)\Big).$$

$$\mathop{\mathrm{L}}_n \mu_n = \theta \quad (\mathfrak{P};\ \mu_0)$$

where $\theta_{ij} = \dfrac{1}{i} \quad (ij)$.

Proof of $\mathfrak{M}^{K_1}$.—Given a function μ in the definition of which

* Cf. § 48*k*1; § 48*m*1. The proof of $\mathfrak{M}^{K_{12*}}$ in this example follows in the same way as proof of § 48*n*1 follows from § 48*k*1 and § 48*l*1.

sequences $\{a_i\}$ and $\{b_i\}$ occur we must show a function μ_0 such that $\mu^{K_1\mu_0}$.

$$(\{a_i\} \,.\, \{b_i\}) :\supset : \exists\, \{c_i\} \ni (i_2 > i_1 \,.\supset.\, Ac_{i_2} < Ac_{i_1}) \,.\, (\mathop{\mathrm{L}}_i c_i = 0).$$
$$(i \,.\supset.\, Aa_i \leqq Ac_i \,.\, Ab_i \leqq Ac_i).$$
$$\{c_i\} \,.\supset.\, \exists\, \{d_i\} \ni (\mathop{\mathrm{L}}_i c_i d_i = 0 \,.\, \mathop{\mathrm{L}}_i d_i = \infty) \,.\, (\mathop{\mathrm{L}}_i a_i d_i = \mathop{\mathrm{L}}_i b_i d_i = 0).$$

$$\mu_1 \ni \mu_{1ij} = d_i \mu_{ij} \,.\supset.\, \mu_1^{B_0 \mathfrak{M}}.$$

The function μ_1 is available as the desired μ_0.

Proof of $\mathfrak{M}^{-K_1\mathfrak{o}}$.—We show that $\mathfrak{M}_* \neq [\text{all } \varphi^{K_1\mathfrak{M}}]$. In the proof of $\mathfrak{M}^{-C}$ we saw that the function θ there defined belongs to $\mathfrak{M}_*$. We will now show that θ is not dominated by $\mathfrak{M}$ and therefore cannot have the property $K_1\mathfrak{M}$. Suppose $\theta^{B_1\mathfrak{M}}$.

$$\theta \,.\supset.\, \exists\, \mu \ni A\theta_{ij} \leqq A\mu_{ij} \quad (ij).$$

$$\mu :\supset : \exists\, i_\mu \ni (i \geqq i_\mu \,.\, j) \,.\supset.\, \mu_{ij} = a_i \cos \frac{j\pi}{2i} + b_i \sin \frac{j\pi}{2i}.$$

$$A\left(a_i \cos \frac{j\pi}{2i} + b_i \sin \frac{j\pi}{2i}\right)$$
$$= \sqrt{a_i^2 + b_i^2}\, A\left(\frac{a_i}{\sqrt{a_i^2 + b_i^2}} \cos \frac{j\pi}{2i} + \frac{b_i}{\sqrt{a_i^2 + b_i^2}} \sin \frac{j\pi}{2i}\right).*$$
$$= \sqrt{a_i^2 + b_i^2}\, A \sin\left(\alpha_i + \frac{j\pi}{2i}\right) \quad -\frac{\pi}{2} \leqq \alpha_i \leqq \frac{\pi}{2}.$$

$$\alpha_i \,.\supset.\, \exists\, j_{\alpha_i} \ni A\left(\alpha_i + \frac{j_{\alpha_i}\pi}{2i}\right) < \frac{\pi}{2i}.$$

$$\theta^{B_1\mu} \,.\supset.\, \frac{1}{i} \leqq \sqrt{a_i^2 + b_i^2} \sin \frac{\pi}{2i} \qquad (i \geqq i_\mu).$$

$$\theta^{B_1\mu} \,.\supset.\, \sqrt{a_i^2 + b_i^2} \geqq \frac{1}{i \sin \dfrac{\pi}{2i}} \qquad (i \geqq i_\mu).$$

$$\mathop{\mathrm{L}}_i \frac{1}{i \sin \dfrac{\pi}{2i}} = \frac{2}{\pi}.$$

$$\exists\, i_1 \ni i \geqq i_1 \,.\supset.\, \sqrt{a_i^2 + b_i^2} > \frac{1}{\pi}.$$

* a_i and b_i are never simultaneously 0. Otherwise μ does not dominate θ. $\sqrt{a_i^2 + b_i^2}$ denotes positive square root of $a_i^2 + b_i^2$.

Thus we have the contradiction that either

$$\mathop{\mathbf{L}}_{i} a_i \neq 0 \qquad \text{or} \qquad \mathop{\mathbf{L}}_{i} b_i \neq 0.$$

It now follows from § 35I; § 20*a*2; § 27.10, that $\mathfrak{M}^{-D_1K_2{}^-K_{12*}}$.

48*q*. *Miscellaneous examples.*—In this section we give examples useful in connection with those already given for the characterizations of the sequel.

(1) $^-LCDA\Delta K_1K_2K_{12*}$; $^-K_{1*}\ K_{2*} : \mathfrak{P}^{\text{II}_2}$; $\Delta_2^{\text{II}_2}$; $\mathfrak{M} \equiv (1, 1)$.
(2) $^-LCDA\Delta K_1K_2K_{12*}$; $^-K_{1*}{}^-K_{2*} : \mathfrak{P}^{\text{II}_3}$; $\Delta_4^{\text{II}_3}$; $\mathfrak{M} \equiv (1, 1, 0)$.
(3) $LCDA\Delta K_1K_2{}^-K_{12*}$; $^-K_{1*}{}^-K_{2*} : \mathfrak{P}^{\text{II}_3}$; $\Delta_3^{\text{II}_3}$; $\mathfrak{M} \equiv (a, a, b)$.
(4) $^-LC^-DA\Delta K_1K_2K_{12*}$; $^-K_{1*}{}^-K_{2*} : \mathfrak{P}^{\text{II}_4}$; $\Delta_2^{\text{II}_4}$; $\mathfrak{M} \equiv (0, 1, 1, 0)$, $(1, 0, 0, 0)$.

49. *Characterizations of classes* $\mathfrak{P}$ *of elements, of developments* Δ, *and of classes* $\mathfrak{M}$ *of functions.*—By comparison of previous theorems and examples there arise many characterizations of classes $\mathfrak{P}$ of elements, of developments Δ, and of classes $\mathfrak{M}$ of functions. Of these characterizations we indicate the principle ones which are evident.

49*a*. *Characterizations* of classes* $\mathfrak{P}$ *of elements.*—

THEOREM. *In the following four sets* (I, II_{12}, II_{123}, II) *of properties of a class* $\mathfrak{P}$ *of elements, any two properties of the same set are equivalent.*

(I 1)	singular.		
(I 2)†	$\Delta^{\square} . \mathfrak{M} . \supset . \mathfrak{M}^{K_2}$.	§ 41.7.‡	§ 48*a*7.
(I 3)	$\Delta^{F} . \mathfrak{M}^{-\Delta^-K_1} . \supset . \mathfrak{M}^{-L}$.	§ 41.6.	§ 48*a*5.
(I 4)	$\Delta^{V} . \mathfrak{M}^{-L} . \supset . \mathfrak{M}^{K_{2*}}$.	§ 41.10.	§ 48*b*5.
(I 5)	$\Delta^{F} . \mathfrak{M}^{-L} . \supset . \mathfrak{M}^{K_{2*}}$.	§ 41.10.	§ 48*b*6.
(I 6)	$\Delta^{\square} . \mathfrak{M}^{-L} . \supset . \mathfrak{M}^{K_{2*}}$.	§ 41.10.	§ 48*b*6.
(I 7)	$\Delta_4^{F} . \mathfrak{M}^{-K_1} . \supset . \mathfrak{M}^{K_{2*}}$.	§ 41.10.	§ 48*a*5.
(I 8)	$\Delta^{V} . \mathfrak{M}^{-K_1} . \supset . \mathfrak{M}^{K_{2*}}$.	§ 41.11.	§ 48*f*2.
(I 9)	$\Delta^{\square} . \mathfrak{M}^{K_{1*}} . \supset . \mathfrak{M}^{K_{12*}}$.	§ 41.8.	§ 48*a*7.
(I 10)	$\Delta^{\square} . \mathfrak{M}^{K_{12*}} . \supset . \mathfrak{M}^{K_{1*}}$.	§ 41.8.	§ 48*q*1.
(I 11)	$\Delta^{\square} . \mathfrak{M}^{\Delta K_1} . \supset . \mathfrak{M}^{K_{1*}K_{12*}}$.	§ 41.12.	§ 48*q*1; § 46*a*7.
(I 12)	$\Delta^{FC} . \mathfrak{M}^{-LK_1} . \supset . \mathfrak{M}^{K_{1*}K_{12*}}$.	§ 41.14.	§ 48*b*5.
(I 13)	$\Delta^{\square} . \mathfrak{M}^{-LK_1} . \supset . \mathfrak{M}^{K_{1*}K_{12*}}$.	§ 41.14.	§ 48*b*8.

* Cf. I. G. A., § 49, for similar characterizations.

† $\Delta^{\square} \equiv \Delta \ni$ every class $\mathfrak{P}^{m\,l}$ is the class $\mathfrak{P}$. I. G. A., § 75.

‡ The references are to the theorem and example which in view of the fact that a composite property existent for a class $\mathfrak{P}_1$ is existent for any superclass of $\mathfrak{P}_1$ (cf. § 48), establish the characterization.

(I 14) $\Delta^{\square} . \mathfrak{M}^{K_1} \neq [0] . \supset . \mathfrak{M}^{K_{1*}}$. § 41.15. § 48*a*6.
(I 15) $\Delta^{FC} . \mathfrak{M}^{K_1} \neq [0] . \supset . \mathfrak{M}^{K_{1*}}$. § 41.15. § 48*b*5.

(II_{12} 1) singular or dual.
(II_{12} 2) $\Delta . \mathfrak{M}^{\breve{}LK_2\breve{}K_{1*}\breve{}K_{2*}} . \supset . \mathfrak{M}^{\breve{}K_{12*}}$. § 42.2. § 48*q*2.
(II_{12} 3) $\Delta . \mathfrak{M}^{\Delta K_2\ K_{1*}\breve{}K_{2*}} . \supset . \mathfrak{M}^{\breve{}K_{12*}}$. § 42.3. § 48*q*2.
(II_{12} 4) $\Delta^{\square} . \mathfrak{M}^{\Delta K_1\breve{}K_2} . \supset . \mathfrak{M}^{K_{1*}}$. § 42.4. § 48*f*6.
(II_{12} 5) $\Delta^{\breve{}V_1F} . \mathfrak{M}^{\Delta\breve{}K_1K_2} . \supset . \mathfrak{M}^{K_{2*}}$. § 42.5. § 48*q*3.
(II_{12} 6) $\Delta^{\breve{}V_1\square} . \mathfrak{M}^{L\Delta} . \supset . \mathfrak{M}^{A}$. § 42.6. § 48*f*6.
(II_{12} 7) $\Delta^{\square} . \mathfrak{M}^{L\breve{}A\Delta} . \supset . \mathfrak{M}^{K_2}$. § 42.7. § 48*f*6.
(II_{12} 8$_i$) $\Delta^{FC} . \mathfrak{M}^{\breve{}DK_i} . \supset . \mathfrak{M}^{K_{i*}}$. § 42.8$_i$. § 48*g*9 ($i = 1, 2, 12$).

(II_{123} 1) singular, dual or triple.
(II_{123} 2) $\Delta . \mathfrak{M}^{\ D\Delta K_1K_2} . \supset . \mathfrak{M}^{K_{1*}} \smile \mathfrak{M}^{K_{2*}}$. § 43.1. § 48*q*4.
(II_{123} 3) $\Delta . \mathfrak{M}^{\breve{}DK_1K_2K_{12*}} . \supset . \mathfrak{M}^{K_{1*}} \breve{\smile} \mathfrak{M}^{K_{2*}}$. § 43.2. § 48*q*4.

(II 1) finite.
(II 2) $\Delta^{F} . \mathfrak{M}^{K_1K_{2*}} . \supset . \mathfrak{M}^{K_{12*}}$. § 44.4. § 48*g*10.
(II 3) $\Delta^{F} . \mathfrak{M}^{\Delta K_1K_2} . \supset . \mathfrak{M}^{K_{12*}}$. § 44.5. § 48*g*10.

49*b*. *Characterizations of developments* Δ.—
(1) $\Delta^{V_1} : \infty : \Delta \ni \mathfrak{M} . \supset . \mathfrak{M}^{\Delta}$.
(2) $\Delta^{F_1} : \infty : \Delta \ni \mathfrak{M} . \supset . \mathfrak{M}^{K_2}$.
(3) $\Delta^{C_1} : \infty : \Delta \ni \mathfrak{M} . \supset . \mathfrak{M}^{K_1}$.
(4) $\Delta^{C_1} : \infty : \Delta \ni \mathfrak{M}^{K_2} . \supset . \mathfrak{M}^{K_{12}}$.

These follow from § 17; § 20*a*2; § 20*a*1 and from the propositions:
(1′) $\Delta^{\breve{}V_1} . \supset . \exists\ \mathfrak{M}^{\breve{}\Delta}$;
(2′) $\Delta^{\breve{}F_1} . \supset . \exists\ \mathfrak{M}^{\breve{}K_2}$;
(3′) $\Delta^{\breve{}C_1} . \supset . \exists\ \mathfrak{M}^{K_1\breve{}K_2}$.

49*c*. *Characterizations of classes of functions.**—
(1) $\mathfrak{M} \equiv [0] : \infty : \mathfrak{M} \ni \Delta . \supset . \mathfrak{M}^{K_1}$.
(2) $\mathfrak{M} \ni \mu . \supset . \mu^{\text{constant}\ (1,\ 1)} : \infty : \mathfrak{M} \ni \Delta . \supset . \mathfrak{M}^{K_2}$.

50. In the Introduction to a Form of General Analysis it is shown that the property K_{12*} is implied by the properties D, Δ, K_1, K_2 and that the properties K_1, K_2 are implied by K_{12*}.† However examples

* Cf. I. G. A., pp. 130.
† § 23*a*17, 11_{12}; I. G. A., § 82II; § 72*a*11_{12}.

given in § 48 of this paper show that the properties D, Δ as properties of classes $\mathfrak{M}^{K_1K_2K_{12*}}$ are completely independent. Many of the cases where the property $^{-}DK_{12*}$ is present are covered by the theorem of § 40I of this paper.

As before remarked we have neither examples illustrating the composite properties,

$$LC^{-}DA\Delta K_1K_2{}^{-}K_{12*}; \quad L^{-}C^{-}DA\Delta K_1K_2{}^{-}K_{12*}; \quad LC^{-}D^{-}A\Delta K_1K_2{}^{-}K_{12*},$$

nor propositions showing that they are non-existent. However, by theorems § 32I; § 35I; § 38I; § 40I, these composite properties are non-existent for certain important and very extensive parts of the general field. Thus we have the following theorem:

THEOREM. *The three composite properties*:

$$LC^{-}DA\Delta K_1K_2{}^{-}K_{12*}; \quad L^{-}C^{-}DA\Delta K_1K_2{}^{-}K_{12*}; \quad LC^{-}D^{-}A\Delta K_1K_2{}^{-}K_{12*},$$

are each non-existent for all systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta^{C_2};\ \mathfrak{M}), \quad (\mathfrak{A};\ \mathfrak{P};\ \Delta^{F_1};\ \mathfrak{M}), \quad (\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M}^{D_1B_1}).$$

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 2 — January, 1913.

(Whole Series, Vol. XVII, No. 2.)

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A Description of the Specimens of the Teleostean Genus Enchodus in the University of Kansas Museum . . *Wyman R. Green.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post office in Lawrence as second class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 2] January, 1913. [WHOLE SERIES VOL. XVII, No. 2

A DESCRIPTION OF THE SPECIMENS OF THE TELEOSTEAN GENUS *ENCHODUS*. IN THE UNIVERSITY OF KANSAS MUSEUM.

BY WYMAN R. GREEN.

Plates I to XVII.

(Contribution from the Zoölogical Laboratory, No. 201.)

A thesis presented to the graduate faculty of the University of Kansas in partial fulfillment of the requirements for the degree of master of arts.

THE purpose of this paper is to offer new facts which will aid in the systematization of the knowledge of *Enchodus*, and, so far as possible with the material at hand, to furnish anatomical data by means of which future specimens may be classified. Since new specimens are constantly being collected, a revision of the classification is necessary from time to time.

After writing this paper the author, in company with Dr. R. L. Moodie, spent five weeks in Logan county, Kansas, making special search for specimens of *Enchodus*. In one stratum of chalk rock about four feet thick, near Elkader, were found several hundred specimens of *Enchodus*, consisting of numerous operculars, dentaries, palatines, etc., of *E. shumardi*, and a considerable number of specimens of *E. dolichus* and *E. amicrodus*. It is the writer's intention to give these consideration in a subsequent paper. No doubt the collection will furnish data for a complete description of the cranial anatomy of *E. shumardi*, and many new facts concerning the other two species represented.

The work was undertaken at the instance of Dr. C. E. McClung, to whom the writer is much indebted for many helpful

Received for publication June 12, 1912

suggestions. He wishes to thank the American Museum also, through Dr. Louis Hussakof, for the loan of their specimens of *Enchodus*. Most of the specimens studied were prepared by the author under the direction of the assistant curator of paleontology, Mr. Handel T. Martin.

All of the drawings were made by the author. Those for figures 3, 4 and 5 of plate II, all of plates III, V, VI, X, XI, XII, XIII and XV were made in the following manner: The specimens were photographed and then carefully traced in India ink, after which the silver salts were removed by means of a thorough washing in a solution of potassium cyanide. This process insures greater accuracy of detail and proportions than is possible otherwise with the most painstaking measurements. Drawings for plates VII, VIII and IX were made direct from the specimens by means of measurements. Drawings for plates XVI and XVII were made by means of the camera lucida.

The literature on *Enchodus* shows that there have been at least sixty-nine different names proposed for the various species of the genus. Thirty-eight of these have been found to be synonymous. Thus about thirty stand as valid species. Since many of these have been based upon fragmentary material, it is quite probable that this number will be much reduced as more complete specimens are found. Except in the posterior regions of the skull, the bones are so loosely articulated that specimens with the elements in position are not often collected. Such parts as the palatines, single teeth, or the cephalic ends of the dentaries are frequently found associated with no other part of the skeleton. This condition, together with the wide distribution of the species and the extreme variability of so many of the characters, as the size of the teeth, their relative proportions, the striations and ridges on the teeth, their number, the angle which the palatine fang makes with the palatine, depth of the dentary, and some others, which have been used in the determination of species, accounts for the large synonymy to be found in *Enchodus* literature.

Specimens have been collected in the United States from Kansas, South Dakota, New Jersey, Delaware, and New Mexico. In all about fifteen species have been described, mostly based on fragmentary material. The genus has also been collected in the Senonian and Turonian of England, Turonian of

Saxony and Bohemia, Danian of Holland, Upper Senonian of Belgium, Upper Cretaceous of Westphalia, Vieux Faugumont of Belgium, Comen beds of Istria, the Isle of Lesina, Dalmatia, Wealden in northern Germany, in southern India, and in the Cretaceous of Brazil.

The great diversity of structure which results from such a wide distribution makes it very difficult to establish definite relations between this fish and other forms. Doctor Loomis regards the presence of teeth on the maxillary, in the form *Enchodus lemoneiri*, as sufficient to give it the rank of a new genus; while Doctor Hay has described a specimen that he considers a new species of *Enchodus* (*E. sævus*), which does not have the row of fine marginal teeth on the dentary. *Enchodus dirus* Leidy and *Enchodus amicrodus* Stewart also lack the row of marginal teeth. This row of fine teeth on the external alveolar border of the dentary is constant in those species which have it, and when absent in a species there is never the slightest traces of it. It seems to be a very definite character. Thus there would seem to be quite as much ground for making a new genus for these two forms as for *Enchodus lemoneiri*. It would not be advisible to do so, however. These facts are offered only to show that much difficulty is met with in fixing the limits of *Enchodus*.

The living families Odontostomidæ and Alepisauridæ are regarded by Woodward (Woodward, 1901) as the nearest living relatives of this extinct genus. *Enchodus* seems to differ from *Cimolichthys* in the possession of a much swollen palatine. Doctor Loomis says, "I regard *Enchodus* as belonging to the same family as *Cimolichthys*, and only as a special form developed in deep water."

The genus has been described by Dr. A. S. Woodward, in his Catalogue of Fossil Vertebrates of the British Museum, in the following words:

> Trunk elongate fusiform, both this and the head laterally compressed. Cranial roof exhibiting a deep median longitudinal depression, its lateral and occipital margins ornamented, like the other external bones, with ridges and tubercles of ganoine. Mandible a little prominent, provided with an inner widely spaced series of large, slender teeth, the largest in front, also a marginal series of minute teeth, all nearly or completely solid; premaxilla in the form of a vertical lamina, deepest in front, tapering behind, and with a single-spaced series of small teeth; maxilla long and slender, either finely toothed or toothless at the oral border; palatine thickened and tumid, with only one large tooth fixed at its anterior end;

ectopterygoid (pterygoid) robust, with a single-spaced series of large, slender teeth, gradually diminishing in size backwards; no teeth barbed. Operculum strengthened on the inner side by a ridge extending horizontally backwards from the point of suspension; branchiostegal rays about twelve to sixteen in number. Vertebræ forty to fifty in number, about half being caudal; the centra at least as long as deep, constricted mesially, and marked with small, irregular, longitudinal ridges. All except the foremost rays of each fin finely divided distally, but none excessively elongated. No post-clavicular plate. Pectoral fins large, pelvic fins much smaller and arising far forwards; dorsal and anal fins large, neither much longer than deep, the former arising much in advance of the middle point of the trunk, the latter also far forwards; a posterior adipose dorsal fin observed in a few well-preserved specimens; caudal fin forked, with curved fulcral rays and stout, articulated, undivided rays at its base both above and below. Rudimentary dermal scutes, not overlapping, in a single median series between the occiput and the dorsal fin, and along the course of the lateral line; a pair of enlarged hook-shaped dermal scutes at the base of the tail, one on either side of the caudal pedicle.

Doctor Woodward states that the cranial osteology of *Enchodus* is best known from the specimens occurring in the English and Dutch chalk, and the trunk and fins are only satisfactorily shown in the nearly complete fishes obtained from the Upper Cretaceous of Westphalia and Mount Lebanon. Fortunately the University of Kansas now has several very complete skulls as well as other parts of the skeleton of *Enchodus,* and we are thus enabled to add our quota to the knowledge of the genus.

The specimens in this collection are at variance with Woodward's description regarding the following characters:

Cranial roof flat, except for the grooved ridges on the lateral margins and in the occipital regions; not all of the species possess a series of minute teeth on the external alveolar border of the dentary; maxillary never toothed; teeth on the pterygoid not always slender, but often quite stout, as in *E. petrosus;* the diminution of the teeth on the pterygoid, caudad, not always gradual, but often the second, third, or even fourth tooth from the cephalic end may be the largest, and the others irregular in size. This irregularity is observable mostly in old individuals. In the young specimens and in the smaller species, as *E. shumardi,* the large teeth of the dentary and the pterygoid diminish in size caudad until they disappear, and usually stand equidistant. The skulls represented in this museum are approximately triangular from a lateral view, the length of the skull being slightly greater

than the length of the mandible, and the depth of the caudal end of the skull somewhat less than the length of the mandible. (See pl. VI, figs. 3 and 4, and pl. XI, figs. 1 and 2.) The head was very large in comparison with the body; hence the fish could not have been a rapid swimmer. The nature of the teeth would indicate that it was a predatory fish, and the fact that of the sixty palatines in the collection only one (pl. XV, fig. 14) shows a new palatine fang replacing the old, would point to the conclusion that *Enchodus* either had few enemies or was very successful in defending itself. Otherwise, it seems, of the forty individuals represented by these sixty fangs, some would have met their death at a time when the new fang had not yet completed its growth. The fangs of both palatine and dentary are well suited to a voracious fish, the palatine fangs being directed cephalo-ventrally at nearly 45 degrees (pl. V, *pal,* and pl. VI, fig. 3) and all of the teeth very sharp. The palatine bears a deep groove on the mesio-dorsal surface (pl. II, fig. 1, *pald*) for the reception of the lateral processes of the ethmoid. These two elements were found in position in *Enchodus petrosus,* specimen No. 836 (pl. IV, *pal, eth*), and are so shown in No. 198 of the American Museum (pls. VIII and IX). The relation of the palatine to the frontal is also shown in the same plate (*pal* and *fr*), and in *Enchodus petrosus,* specimen No. 807, plate V. In the case of specimen No. 836, plate IV, the ventral surfaces of the right palatine and frontal are shown, while in plate V the dorsal surfaces of the left frontal and palatine of the same species are figured.

The palatine is developed in a very unique manner. Numerous writers have discussed the nature of the swollen portion of this bone and have expressed various opinions concerning it: Doctor Loomis (1900) described it as a mass of osteodentine, while Doctor Hay (1903) says that he regards it as "merely very compact bone." Because of these varying and, in this instance, diverse opinions, it has seemed advisable to examine this bone in detail. The results display some very interesting facts. The palatine is quite hard, but may be very easily ground on an emery wheel. Several of the large palatines of *Enchodus petrosus* Cope were used in the investigation. A ventral view of the palatine shows the series of crescentic scars external to the large ventral keel. The keel is well shown in plate VII, figure 15. About the center of each

scar is frequently to be observed a small opening. These openings show much more plainly in some specimens than in others. In many they are closed, and their former positions are indicated by a dark spot on the surface (pl. XV, fig. 15). The line A-O, in plate VII, figure 1, very nearly coincides with a horizontal plane passing through these openings, the cephalic three of which are indicated by the letters *b*, *c*, and *d*. A detailed description of the internal gross structure and development of this interesting element is given in the explanation to plate VII.

The replacement of the palatine fangs has been described substantially in the following words by Doctor Loomis:

> On the anterior end of the palatine there is developed a fold of osseous tissue, which grows downward until it overhangs the palatine fang. This fold is somewhat larger than the base of the senescent fang, and a wing of it extends posteriorly on both sides of the old base. Thus as the new base develops the palatine increases in size. From the ventral margin of this fold the new fang develops.

It has not been determined at just what stage the senescent fang is lost. One of the specimens has the basal portion of the old fang still present, the old fang having been broken off about two millimeters from the palatine (pl. XV, fig. 14). In this specimen it seems that the senescent fang must have persisted until quite a late stage in the development of the new tooth and then to have been broken out. This, as well as the similar cases where two teeth are found standing closely apposed on the ptergoid, would indicate that the swollen bases of the old teeth are not absorbed until after the new tooth is developed. One might expect to find in a large collection of specimens a few that would show the enlarged base of a tooth in the process of absorption. Such specimens are at hand. The conditions of the bases of the discarded teeth vary from a state in which a part of the basal portion of the tooth is still present to almost complete absorption. The absorption process continued throughout a longer period of time than was consumed in the development of three new fangs. This fact is indicated by the portions of old bases which are to be found just caudad to the fang, the ventral surface of the palatine becoming smooth only posterior to the third or fourth scar. The bases of palatine and dentary fangs are never completely absorbed. (See pl. VII, fig. 15, and pl. II, fig. 2.) A dentary, described below, shows very strikingly this slow process of absorption.

Doctor Hay (1903) says there is probably an alternation in the replacement of the palatine fangs. None of the specimens in this collection show this in any marked degree. The fangs do not vary even in the larger species, as *Enchodus dirus* and *Enchodus petrosus,* more than one or two millimeters. This slight inequality comes easily within the limits of individual variation. In no case is there a process of bone extending forward over one fang and not over the other, though several show such processes projecting over both fangs. However, these seem to the author not to be the beginnings of the new fang. A study of longitudinal median sections and coronal sections of the palatine would indicate that they have a different origin. These overhanging processes, which are composed of true bone, are so far above the osteodentine of the palatine that it seems improbable that they have anything to do with the development of the fangs. The palatines which have this process are quite different internally from those which do not possess it Figure 15, plate VII, is nearly a median vertical section of a palatine not having the process, and figure 17 of the same plate is a similar section of a palatine which has it. Certainly this bony process does not grow down and form a tooth of osteodentine. From the shape of the areas, *a, b, c, d,* etc., in figure 17, it would appear that the cephalo-dorsal process over *a,* the base which functioned last, was present over each of the other bases, *b, c, d,* etc., when they were functioning bases, *i. e.,* it was continually present, and that it never was present in the palatine shown in figure 15. Several palatines were sectioned thus by the writer; although not enough evidence can be adduced from the material in hand to make the conclusion positive, the study would indicate that the cephalo-dorsal process and the type of internal structure found in the palatine, shown in figure 17, are constant correlatives. Moreover, it seems to the author that the new bases are not formed by a fold growing down from above, but that there appears over the mesial, cephalic, and to some extent external, portion of the mass of osteodentine a gradually thickening layer of osteodentine, deposited by the odonto-blastic tissue which must have covered these surfaces, the most cephalic portion developing fastest, until the new base was of sufficient size to support the new fang.

Cephalo-dorsal processes occur on the palatines of specimens Nos. 812 and 379 of *Enchodus amicrodus* Stewart, plate X, figures 2 and 3; also in specimens Nos. 831 and 827 of *Enchodus dolichus,* plate XIV, figure 1, and plate XV, figure 11, *pal,* respectively. These processes do not occur on the other specimens. This is somewhat surprising, since Doctor Loomis records having observed, on almost every palatine examined, "a spur of this fold." (Loomis, 1900, p. 275.) The spur referred to certainly can be no other than the cephalo-dorsal process described, since no other similar process is to be observed on any of the other numerous palatines in the American forms represented in the museum of the University of Kansas.

A detailed study of the cephalic end of the palatine suggests a possible explanation of this spur. Note that in the longitudinal median sections, shown in figures 15 and 17 of plate VII, the bases of the fangs (*a, b, c, d,* etc., of both figures) are, at their dorsal ends, somewhat concave in front; but that this is much more marked in figure 17 than in figure 15. Microscopic examination of the cephalic surface of the last functioning base of figure 17 indicates a structure uniformly porous from the surface inward. This would indicate that it does not represent a permanent bone surface, but suggests that considerable of the cephalic portion of each base is absorbed before the deposition of the layer of tissue which will thicken to form the base of each succeeding fang. If this hypothesis is true, the cephalo-dorsal process would not always be equally prominent, nor would the prominence indicate the formation of a new base, but it would be largest at the inception of each new base and most inconspicuous at its completion. Why this absorption should be more marked in some species than in others can not be determined with present data.

The succession of teeth is described by Cope in the following words:

> The first teeth appear on the alveolar border at a considerable distance apart. The second teeth appear immediately in front of these, and by their presence create the irritation which results in the absorption of the root and shedding of the crown of the first. The teeth of the third series appear in advance of the second, occupying the space between them and the empty space previously occupied by No. 1. They may coexist for some time with teeth No. 2, as the specimen indicates, but the result is, as before, the shedding of the adjacent older teeth. In case of the anterior long fang of the dentary, the movement is re-

versed. Here the successional tooth appears behind the functional, which is subsequently shed, and in old fish this tooth occupies a position behind a concave symphysial portion which is edentulous, or only provided with the small teeth of the marginal row.

This description in its entirety applies only to the large teeth of the dentary and pterygoid. And, moreover, the process is not nearly so regular as one might suppose; but the specimens would indicate that only one, or possibly two, teeth on a dentary or pterygoid may be undergoing the process of replacement at the same time. The teeth of *Enchodus* may not properly be said to have roots, as they are anchylosed to the bone. Also, it is found that many of the specimens which show a tooth about to be replaced still have the base, and sometimes the entire old tooth, standing immediately behind the new tooth. However, granting that the process of renewal of the teeth takes place substantially in the manner described, there arise several very interesting questions.

Some of the palatines are found to have on their ventral surfaces as many as fourteen scars of former fangs, and there seems to be no good reason to suppose that the teeth on the dentary and pterygoid are shed much less frequently. There is ample proof that the dentary fang is not shed as often as the palatine fang, yet it seems quite probable that all of the teeth are renewed at about the same rate. Arising, as they do, originally at nearly equidistant intervals, each of the teeth on the pterygoid and dentary (except its fang) is superseded by another which appears immediately in front of it; thus it is very evident that, since the teeth migrate forward approximately by their own diameter at every succession, all of the alveolar surface between the sites will soon have functioned in the formation of teeth. Thus the fourth, or at most the fifth, series will encroach on the territory previously occupied by the first set. Could these old sites have functioned a second time in the formation of teeth?

In case there were fourteen series of teeth on the dentary during the life of the fish, and the teeth had arisen in the first set at such intervals that there was room for three teeth to stand between them, obviously the fourth series of teeth would have to stand on the sites previously occupied by the first set, except the most anterior one (not the dentary fang as it migrates backward), which would stand upon a new

site. At this rate, each bit of alveolar border would function three times during the life of the fish. It may be urged that the dentary is growing all this time and that the increasing size of the interspaces would prevent any complications. Each successive set of teeth is larger than its predecessor, and obviously the ratio of the diameter of the teeth to the length of the dentary would remain a constant throughout the life of the fish. Instead of the growth of the dentary making the problem more simple, it only prevents it from being more complex than it has been stated.

It can easily be seen that each new palatine fang has a distinct base, and no tooth ever appears on any of the old crescentic scars left by previous fangs. Also, in case of the dentary fang, there is never a tooth on any of the scars of fangs which have been shed. Careful examination of all of the specimens of palatines and symphysial portions of dentaries fails to reveal any sign of development of a new fang on a site where there had been a fang standing before. The crescentic scars of the shed palatine fangs extend from the base of the fang to the caudal end of the osteodentine of the palatine, and the scars left by the dentary fang extend from the base of this fang nearly to the cephalic end of the dentary. These portions are always found to be edentulous.

This would suggest a negative answer to the question. There are several specimens of dentaries and pterygoids which may be of some assistance in explaining the nature of this process. A dentary of specimen No. 803 (pl. III, fig. 9, *den*), which shows an especially typical alveolar border, has been refigured in plate II, figure 2, showing the mesial aspect. Unlike the palatine, the tooth scars of the dentary show only on the mesial side. The contour line, *f-f*, shows the relative degree of absorption of the bases of the teeth. The bases of teeth most recently discarded, viz., those immediately caudad to *b*, *c*, *d*, and *e*, are least absorbed; while the older bases, immediately cephalad to *b*, *c*, *d*, and *e*, are most absorbed, but the osteodentine never entirely disappears from the alveolar surface. The latter scars are as large as those just caudad to *b*, *c*, *d*, and *e;* hence they must have been bases of full-sized teeth. Since replacement of the palatine fang commences when the fish is quite small, as shown by the crescentic scars on the ventral surface of this bone, it is reasonable to suppose

that the teeth on the dentary and pterygoid are likewise replaced early in the life of the fish, although no small scars are left as evidence of the fact, as is the case with the palatine. These facts would indicate that the large teeth of the mandibles (excluding the fangs) do migrate cephalad past the original sites of the first series of teeth, and that the scars left by the earlier teeth are obliterated in the process. This is probably accomplished by a migration cephalad of the odontoblasts and enamel cells into the scar areas of the completely absorbed bases, these two classes of cells always remaining grouped in the neighborhood of the bases of the functioning teeth. Quite often the new teeth are inclined cephalad at their bases, due, presumably, to the presence of the old base. (Pl. II, fig. 2, *d*.)

Since the extent of individual variation has not yet been determined, no satisfactory key to the species can be offered at present, but it is hoped that the following may prove helpful. To which group *Enchodus ferox* belongs can not be determined; but because of its large size, it is very probable that it should be placed in the group with *Enchodus petrosus*, although whether it has the row of minute teeth on the external alveolar border of the dentary or not is not known, this species being based upon a single palatine.

Having no marginal fringe of minute teeth on the external alveolar border of the dentary:

Petrosus group.

Palatine fang not situated at the cephalic end of the palatine, but having the palatine projecting forward from the base of the fang more than one-fifth as far as it extends behind it; size as large or larger than *E. petrosus* ..... *Enchodus ferox*

Palatine fang not laterally compressed; an elongate foramen near the mesial border of the premaxillaries; length of palatine fang, 30 to 40 mm. ..... *Enchodus petrosus*

Palatine fang laterally compressed; cutting edge in front only; premaxillaries not as above; a smaller species; length of the palatine fang, 15 to 20 mm..... *Enchodus dirus*

Palatine fang straight and relatively slender; distinctly fluted mesiocaudally; palatine with relatively great vertical thickness; no foramen in the premaxillaries; the smallest species in this group; length of the palatine fang 15 to 18 mm..... *Enchodus amicrodus*

Marginal fringe of minute teeth present on the external alveolar border of the dentary:

Dolichus group.

Smaller than *E. petrosus* and *E. dirus;* palatine relatively slender; palatine fang laterally compressed, and grooved but not fluted as in *E. amicrodus;* length of the fang varies with the size of the specimen, from 7 to 23 mm.....*Enchodus dolichus*

Very small, with bones as thin as paper; length of palatine fang, from 2 to 10 mm.....*Enchodus shumardi*

Enchodus petrosus Cope.

This species was first described by Dr. E. D. Cope in 1874. The description was based upon very fragmentary material. The deplorable custom of hastening to describe a new species upon the discovery of a few bits of bone has resulted in a vague definition of this as well as of other species of *Enchodus.* The best specimens of this species are shown in plates II, III, IV, V, VI, VIII, and IX. It is a large species, but not so large as the unnamed species represented by figures 3, 4 and 5 of plate II. The specimen represented on plate VIII, and in part on plate IX, together with other material, was kindly loaned to the author by the American Museum. The type specimen of *E. petrosus* was included with these specimens. Although Dr. O. P. Hay has described this specimen (pl. VIII) as a new species, *E. sævus,* a careful comparison of all of the specimens has convinced the writer that it is identical with *E. petrosus.* The type of *Enchodus petrosus,* a palatine, has no fang present; but there seems to be no good reason for supposing that this type specimen had a fang compressed differently from the fangs of the numerous palatines, which it so closely resembles, represented in the collection. Since such extensive variation is to be observed in all of the specimens, a more or less distinct cutting edge on a palatine fang, it seems, would not indicate a new species, but only a variant. Nor should the manner of compression in the palatine fang serve alone as a criterion, since considerable variation is apparent here also. Fish bones are not very resistant, and the fangs as well as other portions are found to be twisted, due to uneven pressure of the matrix. The fang shown in plate II, figure 1, *pal,* has undergone torsion, so that the distal end is now compressed almost laterally. The specimen described by Dr. O. P. Hay as *E. sævus* (Hay, 1903) is refigured in plates VIII and IX. Doctor Hay has figured a palatine, preopercular, and what seems to be a premaxillary. Regarding the palatine and the premaxillary there is evidently

an error. He has figured the right palatine complete with its fang, the cephalic end being placed to the right in the figure, and describes it in the text as showing the mesial side. He says, "The figure represents the inner side of the tooth, and shows the inner, or posterior, cutting edge." Further on he describes an outer cutting edge, and the cross section of the fang (fig. 60*a*, l. c.) shows that if one cutting edge is directed outward there can not be an inner cutting edge. To make this description intelligible it is necessary to substitute the word "outer" for "inner" in the two places where the latter occurs in the sentence quoted.

This museum contains two pairs of entire palatines of this species, and with the change suggested the description fits very well. Doctor Hay is evidently in error concerning the palatine and the bones that articulate with it caudally and dorsally in this species. Several of the specimens of *E. petrosus* show the pterygoid firmly attached to the palatine, caudally. There are numerous other species in the collection which show this same arrangement with regard to these two elements.

There are also several specimens which display just such a posterior aspect as is shown in Hay's figures 61 and 62 (l. c.). In view of the fact that so many specimens indicate a firm union of these bones, it would seem that these smooth surfaces must be explained in some other way, perhaps by weathering. The specimen described as an interopercular (fig. 65, *iop*, l. c.) seems to have three rather prominent ridges and other less marked ridges and grooves radiating from a point near the anterior border, and a single row of teeth characteristic of a premaxillary of *E. petrosus*. Since Doctor Hay was in doubt concerning this element, and the author has before him a specimen of *E. petrosus* which has the premaxillary present, so greatly resembling Doctor Hay's figure, he is inclined to think that the specimen in question is not an opercular, but a premaxillary.

Figure 1, plate II, shows an opercular, parasphenoid, palatine and caudal end of the skull of a specimen of *E. petrosus*. It is No. 804. The opercular shows the mesial view, which is smooth except for the rather stout bar extending caudad from the articulation with the hyomandibular; *pal*, *pal'* and *pald* represent a ventral, an external and a dorsal view, respectively, of the left palatine, the most complete of any in the entire collection. The mesial side of this bone sends caudad a

vertical lamina, which probably joins a corresponding lamina on the pterygoid. On the dorsal surface, about one-third its length from the cephalic end, is a deep groove, oblique to the axis of the palatine, in which the lateral processes of the ethmoid articulate. The enlarged caudal end of the parasphenoid is missing. The ventral view shows the angle of compression of the palatine fangs. There are two cutting edges, one directed outward and one cephalad, the mesio-caudal surface being rounded and the proximal half of this surface strongly grooved. The measurements of this specimen are as follows:

	mm.
Cephalo-caudal measurement of opercular	73
Length of the palatine fang	45
Width of caudal end of skull	90
Length of bar on mesial side of opercular	45

Figures 2, 6, 7, and 9, plate III, represent another specimen of *Enchodus petrosus*, No. 803. Palatines of several other individuals are shown on the same plate for comparison. The upper figure of figure 5, and the small specimen, figure 11, may not be *E. petrosus*. The extent of the grooving on the postero-internal surface of the fang is shown in figure 3, specimen No. 833; also just above this tooth is shown a cross section taken about the middle of the fang. The pterygoid (*pt*, fig. 9) is found to be a stout bar bearing nine teeth. One is a very large, laterally compressed tooth, and the others are represented only by bases. The most cephalic of these bases stands at the front end of the bone. Just caudad to this base is another which has begun to be absorbed. The large tooth has a cephalic cutting edge, but the caudal edge is sharp for only about two-thirds of the distance from the distal end. A line drawn in the long axis of the cross section of this tooth would pass external to the palatine fang about twenty-five millimeters. The tooth is grooved mesio-caudally. Caudally the quadrate bears a deep groove for the reception of a cone-shaped process on the cephalic edge of the ventral end of the preopercular. The quadrate has a strong articulation with the articular. The appearance of the mesial surface of a mandible of this species is shown in figure 8, plate III, and the external surface is shown in figure 9 of the same plate. The bases of the teeth caudad to the dentary fang are slightly swollen. This can not be shown, since there is a thin plate of bone along the external alveolar border. There is no fringe

of fine teeth on the dorsal edge of this plate of bone in this species. The base of the dentary fang is very much swollen. Both dentaries display a peculiar mark on the mesial side of the cephalic end, about midway between the fang and the tooth next to it caudally. This is shown in the mesial view of the symphysial portion of the right dentary of this specimen, plate II, figure 2. This has the appearance of a repaired injury, but the fact that no signs of it occur on the external surfaces of these bones precludes this possibility. A similar though not so definite a mark occurs on the dentary of *Enchodus petrosus* Cope, specimen No. 836. The tubercles on the symphysial portion of the dentaries are very much like those on other species. On the externo-caudal side of the base of the fang a small tooth has grown up; also, on the external alveolar border, cephalad to the fang, there were three small teeth situated on an elevated base. The middle one has been broken out, the other two only being shown in the figure (fig. 9, pl. III). The teeth of the dentary and pterygoid are all grooved mesially. The fragmentary premaxillary of this species is irregularly tuberculated (pl. III, fig. 6). The ventral border is thickened, forming a heavy bar. The cephalic end is also thickened, and sends dorso-caudally, at somewhat different angles, two bars, which strengthen the thin dorsal wing. These bars are best observed from the mesial side. The symphysial portion of the left dentary (pl. III, fig. 7) shows the deep mesial groove below the swollen base of the dentary fang. The following measurements of this specimen (pl. III, figs. 2, 6, 7, and 9) may be of service:

	mm.
Length of palatine fang	44
Length of the dentary fang	35
Length of mandible	230
Greatest depth of mandible	50
Length of large tooth on pterygoid	25
Greatest diameter of this tooth at middle	6
From symphysis to nearest point of the articular on the external surface	125

The fine specimen, No. 807, figured in plate V, shows many characters of this species heretofore not known. Most of the cranial elements are represented, though they are fragmentary. The caudal portion of the skull shows the configuration characteristic of this region. The small occipital crest is

present in the median line at the caudal extremity of the skull. The post-temporal, which is so seldom found, lies at the caudal end of the skull, just to the right of the occipital crest in the figure. The bone marked *x* is present in a few specimens, but has not been observed attached to any other bone, and hence has not been determined. It is probably a part of the suspensory apparatus of the pectoral girdle. Fragments of the supraclavicles are to be seen just behind the opercular. They are quite slender and slightly expanded at the ventral ends. The dorsal ends of both are broken off. One of these ends is lying just back of the left side of the skull. Both palatines are present, one of them being refigured (pl. XV, fig. 14) on a larger scale to show the prominent base of a recently discarded fang. The other palatine shows the bases of the discarded fangs absorbed in the usual manner. This is the only indication, among all of the palatine fangs of the specimens of the University of Kansas Museum, of an alternation in the replacement of the palatine fangs; though specimen No. 198 of the American Museum furnishes another instance. The opercular was very large. Its outline is traced with a broken line in the figure. The subopercular is represented, but is only fragmentary. Its impression in the matrix furnishes the outline of this element in position. The interopecular is almost complete, but does not lie in position. There is a small protuberance on the caudal edge of the preopercular about one-third its length from the lower extremity, which is probably for the articulation of the subopercular. There are no teeth on the external alveolar border of the dentary. All of the teeth are slightly grooved mesially. There are thirty-nine vertebræ. Possibly some are missing, but certainly not more than two or three. They are constricted and are irregularly grooved laterally. The caudal fin has been poorly preserved, the rami of the fins being represented only by the proximal ends of the fin rays. The hypural is present and in position. The bone marked *f*, just ventral to the hypural, is an azygous bone, which probably supported a fleshy fin. A much better caudal fin of this species is at hand, and is described below. The specimen under discussion is only about one-half the size of specimen No. 803, and it is quite possible that it is distinct, but since the palatine fangs are clearly of the *Enchodus pet-*

rosus type, it is thought best to figure it as a small specimen of *E. petrosus*. The principal measurements are as follows:

	mm.
Length of vertebral column	550
Width of skull at the sphenotics	75
Width of the caudal end of the skull	55
From the caudal end of the angular to the palatine fang	180
From same point to top of hyomandibular	130
Depth of caudal end of mandible	45
Length of the palatine fang (estimated)	30
Cephalo-caudal diameter of the palatine fang at the middle point	5
Length of the post-temporal	47
Length of the centra	15

These measurements indicate that the fish was comparatively short bodied, with a large triangular head. If the pectoral fins of this species have the same proportional length as those of *Enchodus dolichus* (pl. XI), they are one-half as long as the body.

The caudal fin of this species, represented by another specimen (No. 806, pl. VI, fig. 1), has nearly equal rami. They are comparatively slender and composed of rays which are solid near the proximal ends, but the remainder is segmented, and finely divided distally. The most anterior rays are short, unsegmented bars. These gradually increase in length until about the eighth ray, when they become segmented distally. After the twelfth they elongate rapidly, and the fourteenth is large and heavy, forming a large part of the cephalic border of the rami. The rays which attach to the caudal margin of the hypural are very small and about one and one-half times as long as the hypural. The following measurements were taken:

	mm.
Length of the dorsal ramus	130
Length of the hypural	15

Enchodus sp.

Figures 3, 4, and 5, plate II, represent a fragmentary specimen which may prove to be a new species. It is No. 805. It is larger than any of the other American species yet described. The specimen consists of a portion of an opercular, a nearly complete supraclavicle, the caudal end of a dentary, and some very small fragments of the articular. The opercular was covered externally with radiating tuberculate ridges. The caudal portion of this bone is broken away, but the impression (*im*) left in the chalk shows distinctly the ornamentation of the ex-

ternal surface. The supraclavicle (*scl*) has about the same proportions as that of *Enchodus petrosus* (pl. V, *scls*). Figure 4 shows the fragments of the articular and the caudal ramus of the dentary in position. The broken lines indicate the outlines of the missing portions as represented in the impressions in the matrix. The dorsal ramus is narrow and quite stout. This portion does not send ventrally a thin plate of bone, but is distinctly rounded on the ventral edge, which definitely distinguishes this species from *Enchodus petrosus* Cope and *Enchodus dirus* Leidy. The following measurements indicate its large size:

	mm.
Length of supraclavicle	150
Length of mandible (estimated)	300
Depth of caudal end of mandible	75

Enchodus dirus Leidy.

This species was first described in 1857. The type specimen is a portion of a dentary. A dentary, specimen No. 322 of this museum, was assigned to this species by Stewart (Stewart, 1900). This specimen, though possessing the characters of the type, seems to belong to another species. The characters of the Stewart specimen are: base of dentary fang enlarged; cutting edge begins cephalo-externally at the base, but passes to the front side by the time it has reached the extremity of the fang; mesio-caudal surface rounded and distinctly striated; cross section at the middle point is pyriform; foramen for the blood vessels supplying the fang at the ventral edge of the mesial side of the base; the teeth caudad to the fang slightly striate mesially; they are laterally compressed. A line drawn in the long axis of the cross section of any of these teeth would pass external to the fang. The external surface of the dentary has prominent ridges radiating from the symphysis. This last character was not mentioned by Stewart, and it seems not to have been a character of *Enchodus dirus* Leidy. It seems that all of these characters except the last should be retained for *E. dirus*, but since this dentary can not be distinguished from the dentary of specimen No. 803 (pl. III, fig. 9), which has fangs like the specimen (No. 198 of the American Museum), described by Hay as *Enchodus sævus* (Hay, 1903), it will be referred to the species to which that specimen belongs. Another specimen (No. 86 of this museum), a palatine complete with its fang, described and figured by Stewart as *Enchodus doli-*

chus, can not be identified with either the type specimen of that species, which is now before the writer, or with the specimen which Doctor Hay has described and figured as *Enchodus dolichus* (Hay, 1903). A comparison of this specimen with the type and the other specimens of this species in the American Museum shows it to be at least four times as large as the palatines of that species. While size is not to be depended upon in all cases as a criterion, it seems to be important in this case. Since *Enchodus dolichus* has the external row of fine teeth on the dentary, and this specimen, No. 86, can not be distinguished from the palatine of another specimen in this museum which does not have the fine fringe of teeth on the external alveolar border of the dentary, it will be classed with the latter. The specimen referred to is No. 817. The dentary of this specimen possesses all of the characteristics that are given for *Enchodus dirus*. Whether it is the same as the type or not will matter little, since a number of species have the same characters as those possessed by the type specimen of *Enchodus dirus*, and it is hoped that this specimen will have enough definite characters to establish the species in a way that will facilitate the work of future systematists, instead of being a hindrance, as is always the case when new species are founded upon insufficient material. This specimen (pl. VI, figs. 3 and 4) differs from that described as *Enchodus sævus* by Doctor Hay, in not having any ridges on the external surface of the dentary; only slight traces of tuberculated ridges on the external surface of the articular; palatine fang laterally compressed and pyriform in cross section; the fang has only one cutting edge, which is situated in the median line in front. Other characters are: distinct grooves on the mesio-caudal surface of the fang; a deep groove on the premaxillary, beginning 3 mm. from the alveolar border and as far from the mesial edge, extending caudally, gradually drawing near the median line, so that 20 mm. from the cephalic end of the premaxillaries the grooves are only 1 mm. from the mesial edge of the bone. The remainder of the external surface is neither ridged nor tuberculated. The premaxillary teeth are 2 mm. long, very short for a specimen of this size. The premaxillaries are one-half as long as the mandibles. The ethmoid (fig. 3, *eth*) is composed of two stout beams, which are enlarged and firmly united at the cephalic ends, which, in turn, are flexed abruptly outwards, forming a pair of short processes, the long axis of the cross

section of which cuts the long axis of the beams at an angle of 45°. Since the caudal edge of these processes is ventral, and the grooves on the dorsal side of the palatines are very nearly at right angles to the palatine fangs, it is plain that when the processes are in place in the grooves on the palatines the long axis of the fangs will make an angle of 135° with the long axis of the ethmoid, and hence the dorsal surface of the skull, since these coincide. In plate VI, figure 2, *eth*, the broken line from *eth* points to the base of the process which articulates with the right palatine. The maxillary (fig. 3, *mx*) is shown almost complete in the same figure. It is enlarged at the cephalic end and sends a process upwards, which in all probability met its fellow of the opposite side in the median line. The caudal half of this bone is a slender bar reaching to the caudal end of the dentary. The palatine could not be distinguished from that of *Enchodus petrosus* were it not for the fang, which has a different shape. The palatine and dentary fangs are bent forward at the apex. The teeth of the pterygoid and dentary are all flexed backward about one-third their distance from the bases, and forward again near their apices. This bending is slight, but easily observable. The metapterygoid and mesopterygoid are represented. The former is shaped much like that of *Enchodus amicrodus* (pl. X, fig. 3). The preopercular is expanded at the ventral end and tapers to a point at the dorsal end. The ventral end is marked distinctly by radiating ridges. *Enchodus dirus* differs from *Enchodus amicrodus* in not having distinct tuberculated ridges on the external surface of the dentary; the ratio of the diameter of the palatine fang to its length is more than twice as great as that of *Enchodus amicrodus*, i. e., the palatine fang of *Enchodus amicrodus* is only one-fourth as stout as that of *Enchodus dirus*, the fang of the former being slender, straight, and fluted mesio-caudally.

The preopercular is in position. The hyomandibular fits into a deep groove on its cephalic border. There is a cone-shaped process on the cephalic border near the ventral end, which fits into a deep groove on the caudal edge of the quadrate. Only the cephalic edge of the opercular is present. The subopercular is almost complete. It has numerous ridges radiating from the point of articulation with the hyomandibular. The relation of the premaxillary to the palatine is well shown in the left view of this specimen. (See pl. VI, fig. 4.) The

swollen portion of the palatine lies in the concavity on the mesial surface of the premaxillary. There is a bone shown in the left view (fig. 4, *c*) which may be a portion of the ceratohyal. It is narrow at the ventral end and gradually widens dorsally. About one-third the length from the ventral end it divides into two parts. While passing under the ventral margin of the articular it splits into eight narrow strips. These are somewhat broken. Their dorsal ends extend under the caudal end of the dorsal ramus of the left dentary. Fifteen branchiostegal rays are present. The species is larger and the bones are relatively much stouter than in *Enchodus dolichus*, and it is unlike *E. dolichus* in having no fringe of minute teeth on the external alveolar border of the dentary. The following measurements were taken:

	mm.
Length of mandible	185
Length of maxillary	110
Length of preopercular	105
Length of premaxillary	75
Length of dentary fang	12
Length of palatine fang (estimated)	17
Greatest diameter of palatine fang at middle	6
Greatest diameter of dentary fang at middle	4.5
Depth of caudal end of mandible	45

Enchodus amicrodus Stewart.

The type of this species is a single dentary, described and figured in the University Geological Survey of Kansas, vol. VI, pp. 378-379. The characters given are as follows: absence of the row of minute teeth on the alveolar border of the dentary; the dentary is shallow, and supports nine large teeth, the cephalic one being slightly recurved at the apex; and the "base does not seem to be expanded" as in some of the other forms; external surface of the fang not striated. This species was represented by the type specimen only, which, unfortunately, has been lost. However, numerous other specimens have been found since which appear to be the same species, and from these it is to be hoped that the species may be put upon a firm basis. It seems best to make a cotype of the most complete of these specimens. This one is shown in plate X, figure 3. It is No. 379.

Such differences as are found between the skull of this species and that of *Enchodus dolichus*, in so far as the speci-

mens at hand indicate, could easily fall within the limits of normal variation. The contour of the skull, as indicated by line *c-c*, taken at the line *s-s*, is very similar to the contour of *Enchodus dolichus*, at the same place (pl. XI, fig. 1). The absence of the minute fringe of teeth on the external alveolar border of the dentary is the definite character which separates this species from *Enchodus dolichus* and *Enchodus shumardi*. It is about the size of the former, and the skull is about twice the length of the skull of the best specimen of *Enchodus shumardi* (pl. XV, fig. 1). The palatines are very nearly in position, but the fangs are turned outward. Both display the mesial aspect. The deep groove in which the ethmoid (*eth*) articulates is shown in the right palatine just cephalad to the letters *pal*. It runs very nearly parallel to the ventral surface of the palatine, and its cephalic end is in line with the palatine fang. This groove and its fellow of the opposite bone receive the short lateral processes on the cephalic end of the ethmoid. The latter is complete in this specimen and is represented in the figure. The relation of the ethmoid to the palatines is better understood from inspection of plate VIII, where a dorsal view of the corresponding elements of one of the larger species is represented. This figure shows these bones in position. The left palatine of the cotype of *E. amicrodus* has the fang broken off at the middle. This palatine is attached to the pterygoid caudally. The palatine fangs are very straight, slender, and fluted mesio-caudally. They are compressed laterally, with a cutting edge in front. The short cephalo-dorsal processes over the bases of the palatines are at unequal distances from the base of the fangs, the process on the right palatine being four millimeters and that on the left palatine six millimeters from the base of the fang. The dentary fang is distinctly grooved mesially. The other teeth are only slightly grooved on this surface. The pterygoid teeth have their external surfaces slightly striated. The dentary is marked externally with definite rows of small tubercles, which radiate from the symphysis. That the symphysial portion of the dentary was tuberculated also is shown by the impression of the left dentary in the matrix. The metapterygoid and mesopterygoid are both represented. The right metapterygoid is complete but not in position. The left metapterygoid, which was exposed to view by removing a por-

tion of the chalk from the opposite side of the specimen, is in position relative to the left pterygoid (*pt'*). The position of the left metapterygoid, underneath the right pterygoid and caudal end of the left dentary, is indicated by the broken line which encloses an area in the figure the same shape and size as the right metapterygoid (*mt*). The mesopterygoid is also in position relative to the metapterygoid and pterygoid. The articular is characterized by definite rows of small tubercles radiating from the cotylus. The shape of the preopercular is shown in plate X, figure 1, *pop.* No radiating ridges are to be seen on the expanded portion of this bone in any of the specimens, but they were probably present, as so many of the other bones display this character. This element is poorly shown in the cotype. The left ceratohyal (*ch*) is fragmentary, displaying no definite character. A portion of the right ceratohyal (*ch'*) is present. It lies just to the right of the opercular in the figure. The epihyal is attached to the caudal end of it, and lies beneath the right metapterygoid. The broken line across the lower part of the right metapterygoid indicates the outline of the dorso-caudal edge of this element. (Cf. pl. XIII, fig. 1, *eh;* pl. XIII, fig. 3, *eh;* and pl. XI, fig. 2, *eh.*) The opercular is strongly marked externally with radiating tuberculated ridges. None of the specimens of this species shows an external view of an opercular, but the specimen represented in plate X, figure 1, shows the impression of the external surface of both the opercular and the subopercular, in the matrix. This indicates that they were both decorated with distinct rows of tubercles radiating from the points of suspension. The ventral margin of the opercular is concave, the curvature being quite regular. This margin is present in both figure 1 and figure 3. The mesial surface is smooth, except for the stout bar which is present in all of the species. The opercular is in position relative to the subopercular in figure 1. The interopercular is also decorated with numerous tubercles, but they are very irregularly arranged. The suspensory pharyngeal (*s*) is lying across a preopercular just to the right of the metapterygoid in the figure of the cotype. This bone is equal in length to the width of the ethmoid just caudal the cephalic processes. The larger end has nearly twice the diameter of the smaller, and the middle is slightly con-

stricted. The diameter of the larger end is equal to the greatest diameter of the palatine fang near the base. The quadrate is very similar to that of other species. The basioccipital and the exoccipitals share equally in the formation of the occipital condyle.

Measurements:

	mm.
Width of skull across sphenotics	45
Width of skull across pterotics	39
Width of the shallow depression on dorsal surface of the skull along the line *s-s*	17
Width of the occipital condyle	8
Greatest depth of the palatine	11
Greatest diameter of the palatine fang at middle	2.5
Length of the palatine fang	18
Greatest width of the ethmoid	12
Length of the dentary	90
Estimated depth of the dentary	25
From the articulation of quadrate to palatine fang	90
Length of the articular	65
Greatest depth of the articular	22
Length of the suspensory pharyngeal	8

Specimen No. 51, represented in plate X, figure 1, shows the teeth of the right dentary, except the fang, nearly complete. The second and fourth of these have their points broken off. The external surface of the dentary is exposed. Most of the symphysial portion is missing, but the impression left in the matrix indicates that the fang was strongly grooved mesially. The opercular and subopercular have retained their position relative to each other, but have been moved backward and upward from their position relative to the preopercular. Their characters have received attention in the discussion above of the cotype (No. 379). The dentary has been reversed.

Measurements:

	mm.
Length of the dorsal edge of the dentary	85
Length of the dental fang	13
Length of third tooth caudad to the fang	8
Depth of the mandible at the caudal end	27
Caudo-cephalic measurement of the opercular	37

Specimen No. 812, shown in plate X, figure 2, displays the mesial view of the alveolar border of a dentary, a pair of palatines without fangs, a pair of pterygoids, and a pair of quadrates. The bases of the teeth are swollen much the same as in other species. The statement in Stewart's description of the type, that the base of the dentary fang "does not seem

to be expanded," is easily explained by the fact that only the external surface of the type was exposed, and the bases of the teeth of the dentary can not well be observed from the external view, because the outer margin of the alveolar border is elevated, forming a thin lamina of bone. (See pl. X, fig. 3, *den*). The cephalo-dorsal processes over the bases of the fangs are at unequal distances from the bases of the fangs, and the process on the left palatine is considerably larger. The process over the left fang is four millimeters from the base, and that of the right palatine is five. The larger size of the process on the left palatine may be due to the absorption of the osteodentine of the cephalic end, between the process and the base of the fang, preparatory to the deposition of osteodentine that is to form the base of a new fang. This explanation is suggested by the appearance of a median section of a larger species, possessing a similar process over the base of the fang (pl. VII, fig. 17). The grooves on the mesio-dorsal surface of these palatines for the reception of the lateral processes of the ethmoid are somewhat larger and longer than in the cotype. The figure represents the palatines relatively more slender also, but this is due to the position they occupy and to some extent to compression.

Measurements:

	mm
Depth of the palatine at the middle of the dorsal fossa for the articulation of the ethmoid	9
Width at the same place	7

Other fragmentary specimens of this species in the collection are: No. 813, two dentaries; No. 55, two dentaries and one articular; No. 53, portions of toothed elements, probably dentaries; No. 811, an articular and dentary.

Enchodus dolichus Cope.

This specimen is represented in the collection by thirteen specimens, figured in this paper, and one pair of dentaries, No. 815, not figured. It is distinguished from the *petrosus* group by the presence of the fringe of minute teeth on the external alveolar border of the dentary, and from *Enchodus shumardi* by its much larger size, opaque bones, and the teeth relatively less slender. After a careful comparison of all of these specimens with the type specimen of *Enchodus dolichus* Cope, No. 1820, and with the specimen of *Enchodus dolichus*,

No. 1837, and *Enchodus gladiolus* Cope, No. 1818, all three of which are in the American Museum, I am convinced that they are of one species. The only definite point of difference that can be observed is the more prominent grooving of the palatine fang of the *Enchodus gladiolus* specimen. That such a character as this does not merit so much consideration is proved by the fact that equally as great variations as these are constantly met with in the study of the other specimens. The two large palatine fangs represented in plate III, figure 1, are from different specimens of the same species but the grooves on the right palatine are very prominent while those on the left palatine are very low and broad. They have all the appearance of having been worn down. The compression in each of the fangs is identical, being much the same as in the American Museum specimen, No. 198, described as *Enchodus sævus* by Doctor Hay (Hay, 1903). Such differences as these may be explained in part by the difference in the age of the individuals, but probably to a greater degree by the difference in the age of the teeth. Aside from this, there is great normal variation observable in the genus which is assignable to no definite cause. An illustration of this is to be seen in the pair of fangs referred to. The left palatine (upper figure) has two distinct grooves on the mesio-dorsal surface, the caudal one of which is for the articulation of the ethmoid. The two grooves are separated by a ridge of bone (*r*). No such ridge occurs in the right palatine (lower figure), and the groove is as large as that of the left bone would be if the ridge were not present. Such dissimilarities are frequent even in paired bones. In plate III, figure 2, there is to be seen a depression on the cephalic end of the right palatine just above the base of the fang. It could not be determined to which species several of the specimens in this collection belong on a basis of the grooves on the palatine fangs, as this character varies from almost imperceptible striations to grooves as distinct as those found on the palatine fang of the specimen of *Enchodus gladiolus*, No. 1818, of the American Museum. This difference alone is wholly insufficient as a specific character.

The best specimen of *Enchodus dolichus* in the collection is No. 814, figured in plate XI. It consists of a very complete skull with six vertebræ attached, and a fine pair of pectoral fins. The dorsal surface of the skull is so much like that of

Enchodus amicrodus that without other elements present it would be difficult to distinguish them (cf. pl. X, fig. 3). The palatine fangs are not shown in the figure. They are covered by the premaxillaries, but one fang may be seen in the specimen. It is broken near the middle, showing it to be laterally compressed, with a cutting edge in front. The premaxillary is well preserved. The alveolar border of it is three-fourths as long as the alveolar border of the dentary. The premaxillary teeth of this species are relatively more than twice as long as those of *Enchodus dirus* (cf. pl. VI, fig. 3). The dentary has a few ridges radiating from the symphysis; the articular has numerous slightly tuberculated ridges radiating from the cotylus. The minute teeth on the external alveolar border of the dentary are one-half as long as the premaxillary teeth. In this specimen this row of minute teeth on the dentary is not interrupted as it passes the dental fang. The small teeth on the symphysial portion of the dentary are broken off, but their positions are easily determined by the bases, which are plainly visible (fig. 1, *den*). This is true of the specimens figured in plate XIII, figure 2; plate XIV, figure 2; and plate XV, figure 10. But the dentary, No. 57, figured and described by Stewart as *Enchodus dolichus* (Stewart, 1900), has no teeth external to the base of the dental fang, there being a space of five millimeters where there are no small teeth. The pair of dentaries, No. 815, not figured, has two small teeth external to the base of the left fang. Thus this character seems to be subject to considerable variation in this species, and has little or no value as a specific character. The opercular has many large, irregular ridges radiating from the point of attachment to the hyomandibular. These ridges become more regular, smaller, and tuberculated, on their distal two-thirds. Nearly the whole of the subopercular is covered with small, even-sized, tuberculated ridges radiating from the point of attachment to the opercular. These become less numerous, larger, and less tuberculate near the dorsal and ventro-cephalic edges of the bone. A small area, averaging five millimeters in width, along the ventro-cephalic edge of the bone is smooth. The stout, tapering bar on the mesial side of the opercular is shown in figure 2. The epihyal (fig. 2, *eh*) is very thin at the cephalic end, and thickens caudally. On the caudo-dorsal edge is a knob for the articulation of the interhyal. This latter element

has not been observed in any of the specimens. The epihyal is represented in several of the other specimens: plate XIII, figure 1, *eh;* also figure 3, *eh;* and in the American Museum specimen, No. 1837. In Doctor Hay's description of this specimen he speaks of this element as an opercular (Hay, 1903). There were then no specimens of this species, with the opercular present, in the American Museum. The ceratohyal (pl. XI, fig. 2, *ch*) is lying in position relative to the epihyal. With the exception of the hyomandibular (fig. 1, *hm*), the remaining elements of the hyoid series have not been observed in any of the specimens. The left post-temporal is lying just caudad to the skull, not quite in position. A portion of the supraclavicle is attached to the caudal end of this bone. The other post-temporal, shown in figure 3, is attached to the dorsal end of the supraclavicle (*scl*). In figure 1 the left post-temporal seems to have been moved from its natural position toward the median line of the skull. In all probability the narrow cephalic end should lie directly over the left epiotic, and the slender process (fig. 1, *p*) should attach to the mesial surface of the left pterotic. Since no supratemporal has been observed in so complete a specimen it would seem that this element may not occur in *Enchodus dolichus.* The supraclavicle is not quite complete in this specimen. (A complete one is shown in plate XIII, figure 1, *scl*). This bone attaches to the dorsal process of the clavicle (fig. 3, *cl*). It probably overlapped with the clavicle about one-half its length. The length of these suspensory bones indicates that the pectoral fins were very close to the head, and that *Enchodus dolichus* was not so deep in the region of the pectoral fins as *Enchodus petrosus.* (Compare the supraclavicles of this species with those of the specimen shown in plate V, *scls.*) Plate II, figure 3, shows the supraclavicle of an undetermined species. This is much longer relatively than the supraclavicles of *Enchodus dolichus.* The pectoral girdle and fins are very complete. Plate XI, figure 3, shows the dorsal aspect with all parts below the supraclavicles in position. The shape of the clavicle is best seen in another specimen (pl. XIII, fig. 3, *cl*). The hypercoracoids (*hc*) are complete. They are semilunar in outline, the straight edges having been directed caudo-dorsally in life. The ventral edge joins the hypocoracoid and the dorsal wing is free. Near their ventral border is a large oval foramen. Caudo-dorsally from

the foramen, on the somewhat thickened edge, is a facet for the articulation of the first large pectoral ray. The cephalic edge unites firmly to the mesocoracoid and the clavicle. A prominent ridge begins just dorsad to the foramen and proceeds nearly cephalad across the suture between the hypercoracoid and the mesocoracoid, and, gradually diminishing, disappears at the cephalic edge of the mesocoracoid. The cephalic end of this ridge is as near to the median line as is the mesial edge of the large oval foramen of the hypercoracoid. The mesocoracoid (*mc*) is a much smaller bone, which joins the hypercoracoid on the cephalic edge, and its dorsal edge is firmly united to the clavicle along the line of the ridge mentioned above. The ventral edge is free. The hypocoracoids are both slightly broken at both ends. A more complete one is shown in plate XIII, figure 1, *hco*. The figure of the specimen under discussion shows, by means of broken lines, the outlines of these bones, as determined by their impression in the matrix. The ventral margins of these elements are flexed abruptly outward and then inward, which gives them the appearance of having a groove along the mesial edges. These edges are extended cephalad, ending in narrow points near the median line. The caudal ends are broad, rounded, and directed slightly away from the median line. The middle portion of each hypocoracoid is raised, forming a very flat cone with a rounded apex. Two-thirds of the distance from the apex to the cephalic border in each bone is an oval foramen whose diameter is equal to one-third of the diameter of the foramen in the hypercoracoid. Three actinosts are present. These articulate with the margin of the hypocoracoid just caudad to the hypercoracoid. The fin rays are finely divided distally and are wholly unsegmented.

The measurements of this specimen are:

	mm.
Length of skull	150
Width of skull across sphenotics	70
Width of skull across pterotics	60
Length of the mandible	145
Length of the minute teeth of the dentary	2
Length of the premaxillary teeth	4
Caudo-cephalic measurement of the opercular	92
Estimated length of the clavicle	90
Length of the hypocoracoid	90
Length of the hypercoracoid	30
Width of a pectoral fin at the middle	80

A much less perfect specimen of this species is No. 174, represented in plate XII, figure 2. It is about two-thirds as large as specimen No. 814, discussed above. The palatine and dental fangs are plainly grooved mesially. The large teeth of the dentary display almost imperceptible striations. The palatine fangs are relatively more slender than those of specimen No. 814. There is a cephalo-dorsal process, very similar to that of the specimen shown in plate VII, figure 17, over the bases of the palatine fangs. The dentaries appear to be relatively deeper than in the specimen described above, but this is due to the fact that these dentaries are nearly complete, while in the above specimens both dentaries have their ventral margins broken off, and the right one is considerably crushed dorso-ventrally. Unfortunately both opercular present the mesial surface. They are very thin at the edges, so that the ridges on the external surface show through. The cephalic end of the subopercular is present, showing the numerous small radiating ridges on the external surface. The parasphenoid is more nearly complete in this specimen than in any of the others in the collection. There is a deep median dorsal groove near the broad caudal end. This gradually disappears as it approaches the middle of the bone, and is replaced on the cephalic half by a distinct median ridge. The bone is narrowest at a point near the middle, gradually becoming broader cephalad. It is just twice as wide at the cephalic end as it is in the middle. The bone marked *mt* can not be certainly identified. It differs from the metapterygoid of specimen No. 814 in having the concave portion of the margin not a part of a circle, as is the case with that specimen. This bone can not be certainly identified until more specimens have been collected. If it should be a metapterygoid this specimen could not be *Enchodus dolichus*. However, it seems best not to establish a new species for this specimen so long as there is any possibility of it belonging to *Enchodus dolichus*.

	mm.
Length palatine fang	12
Length dentary fang	9
Length articular	55
Length of alveolar border of dentary	60
Length of parasphenoid (estimated)	60
Length of clavicle	55
Greatest depth of dentary	17
Greatest depth of articular	18
Cephalo-caudal measurement of opercular	33

The dentary shown in plate XII, figure 1, specimen No. 820, has a relatively shorter fang, the other large teeth much closer together, and the stout mesial bar-like portion just ventral to the alveolar border, proportionally much deeper dorso-ventrally than the corresponding part of the dentary in specimen No. 174. This last characteristic appears to be due to compression, and the arrangement of the teeth is probably only a variation.

Measurements:

	mm.
Length of the fang	15
Average diastema	4.5

Specimen No. 831, shown in plate XIV, figure 1, represents an individual which was about one-fifth as large as the specimen first described, No. 814. Only the left palatine, the right articular, the caudal end of the skull, and an undetermined bone, represented. The palatine shows the cephalo-dorsal process over the base of the fang. The ventral surface of the skull is exposed.

Measurements:

	mm.
Length of palatine fang	7
Length of articular	44
Greatest depth of the articular	15
Width of the skull at the sphenotics	30
Width of the caudal end of the skull	25

Specimen No. 821, shown in plate XIV, figure 2, is about one-third larger than No. 831. It is very fragmentary. Enough of the proximal end of one of the palatine fangs is present to show that they were distinctly grooved.

Measurements:

	mm.
Greatest depth of the articular	19
Length of alveolar border of dentary	65
Length of dentary fang	10
Length of premaxillary	39

Specimen No. 808, shown in plate XIII, figure 1, is disarticulated, and thus displays clearly the shape of several elements not so well exhibited in the other specimens. The supraclavicle, epihyal, subopercular, hyper- and hypocoracoids, are very nearly perfect. The ethmoid is attached to the frontals caudally. The bone labeled *bs* has its upper concave surface

pitted as though it had borne many small denticles. It is probably one of the basibranchials. The vertebræ are irregularly grooved laterally and constricted mesially. The proximal third of both pectoral fins is present, but dissociated. The laterally compressed palatine fang is lightly striated mesio-caudally and has the cephalic cutting edge unusually sharp. The caudal end of the skull is present, but badly crushed. The small supra-occipital crest is preserved entire.

Measurements:

	mm
Width of the caudal end of the skull	60
Length of the supraclavicle	70
Length of clavicle	75
Length of palatine fang	23
Length of alveolar border of the dentary	120
Length of hypocoracoid	80
Length of centra	15
Length of the dentary fang	17

Specimen No. 816, plate XIII, figure 2, represents a pair of mandibles, fragments of the palatine, pterygoid, and quadrate. It displays no dissimilarity from specimen No. 814, described above, except that it is somewhat smaller.

Length of the mandible ..... 135 mm.

Specimen No. 819, plate XIII, figure 3, is disarticulated and very fragmentary. The epihyal (*eh*) is complete. This specimen is the same size as specimen No. 808, plate XIII, figure 1.

Specimen No. 810, plate XIII, figure 4, represents a pair of palatines, a pterygoid, and a quadrate. Only one of the palatine fangs is present, exposing the outer surface, which is quite smooth.

Measurements:

	mm.
Distance from the caudal edge of the quadrate to the cephalic end of the pterygoid	80
Longest tooth on the pterygoid	12
Length of the palatine fang	16

Specimen No. 832, plate XV, figure 12, represents a pair of complete palatines, a premaxillary, and a fragmentary left dentary. These remains are from a much smaller individual than those just considered. The fangs are grooved on the caudal surface. The specimen is unique in having the dental fang as long as the palatine fang. The dental fang is distinctly grooved mesio-caudally, and the other large teeth of the dentary are minutely striated mesially. The small teeth on the

external alveolar border of the dentary are relatively longer and more slender than usual.

Measurements:

	mm.
Length of palatine fang	8
Length of premaxillary	30
Length of alveolar border of the dentary	52
Length of dentary fang	8

No. 827, plate XV, figure 11, is a fragmentary specimen, consisting of a pair of palatines, pterygoid, subopercular, portion of a preopercular, a premaxillary, a quadrate, and the symphysial portions of the dentaries. This specimen has the caudal end of the pterygoid complete and not attached to the quadrate. It has on the dorsal surface, five and one-half millimeters from the caudal end, a very thin lamina extending dorso-caudally. The premaxillary is also complete. It is unique in possessing on the external side a row of very minute processes, which appear to be denticles. The row begins three and one-half millimeters from the cephalic end of the bone and one millimeter from the alveolar border, and extends caudally parallel to this edge. The row is three millimeters long. This character has not been observed on any of the other specimens of *Enchodus*. Whether it has any specific value can not now be determined. Since the specimen is small, and possibly a young individual, it is thought best not to attach much importance to the presence of these inconspicuous processes. The specimen may prove to be distinct, but it has been considered here because the other characters would indicate that it is *Enchodus dolichus*. The dentaries have the characteristic external row of minute teeth and the ridges radiating from the symphysis. The palatine fang is somewhat more slender and straighter than the fang of the American Museum specimen No. 1837, described by Doctor Hay in 1903.

Measurements:

	mm.
Length of the palatine fang	10
Length of the dentary fang	7
Length of the premaxillary	27
Greatest depth of the premaxillary	6.5

Specimen No. 156 is a single mandible, which is represented in plate XV, figure 10. It differs from the mandible of specimen No. 174 in having a few more radiating ridges on the external surface of the dentary. Only four large teeth remain

caudal to the fang. The row of minute teeth is continuous past the fang.

Measurements:

	mm.
Length of the mandible	57
Length of the dentary fang	9
Length of the four other large teeth	4.5
Depth at the caudal end of mandible	12

The following table gives the comparative lengths of the palatine fangs of several of the specimens of *Enchodus dolichus*. Although the ratio of the lengths of the fangs to the other measurements varies, the variation is within narrow limits, so that the sizes of the specimens are nearly proportional to the lengths of the fangs.

	mm.
No. 831, plate XIV, figure 1	7
No. 832, plate XV, figure 12	8
No. 821, plate XIV, figure 2	10
No. 827, plate XV, figure 11	10
No. 174, plate XII, figure 2	12
No. 810, plate XIII, figure 2	16
No. 808, plate XIII, figure 1	23
No. 814, plate XI, figures 1, 2, and 3, (estimated)	23

Enchodus shumardi Le dy.

Enchodus shumardi, the smallest species of the genus, is represented in the collection by thirteen specimens. The best of these is the skull shown in plate XV, figure 1, specimen No. 829. This specimen shows the shape of the head, the protruding mandible, and several other characters heretofore unknown. The premaxillary, ceratohyal, branchiostegal rays, and ethmoid do not lie in position. The ethmoid lies four millimeters directly above the deep groove, on the dorsal side of the palatine, in which it articulates. Only a slender portion of the margin of the left frontal remains to connect the ethmoid with the caudal elements of the skull. The premaxillary is reversed.

The palatine in only three-fourths as long as the palatine fang. The pterygoid in this specimen has no teeth. It is very probable that the tooth border has been broken off of this bone, but this can not be determined. The bones are very delicate, and excepting such parts as the palatine, alveolar borders, and points of articulation, are as thin as paper. The ventral border of the dentary is extended into a thin plate, the cephalic end of which is just under the palatine fang.

This character is not to be found on any of its other specimens. It is probably only a variation. The fringe of fine teeth on the external alveolar border of the dentary is present. The small size is shown by the following measurements:

	mm
Estimated length of the skull	50
Length of the mandible	38
Greatest depth of the mandible	10
Length of the palatine fang	7
Length of the dentary fang	6

Specimen No. 823, represented by figure 2, plate XV, furnishes only one valuable character, the shape of the ventral end of the preopercular. The dorsal end is broken off. The other two bones are of no value.

	mm.
Greatest width of the preopercular	9

Figure 3, plate XV, shows a complete dentary. The specimen number is 824. The ventral border is quite different from that of figure 1, being less prominent. Measurements are as follows:

	mm
Greatest depth	7
Length of dentary fang	3
Length of alveolar border	27

Figure 4, plate XV, specimen No. 321, shows the dentary described by Stewart as *Enchodus parvus*. (Stewart, 1900, p. 192.) The mesial aspect is shown. By removing the chalk from the other side the external row of fine teeth was disclosed.

	mm
Length of the alveolar border	25
Length of the dentary fang	2.5

Figure 5, plate XV, is a mesial view of another fragmentary dentary, having a crooked fang. The specimen number is 825. The shape of the fang is probably due to compression. The external row of fine teeth is well shown on the caudal half of the tooth border, but they have been broken off on the cephalic half. Only the fang and one other of the large teeth are present. The fang is strongly grooved mesially. The fine teeth are unusually long.

	mm.
Length of tooth border (estimated)	39
Length of dentary fang	7
Length of the fine teeth	1.5

Figure 6, plate XV, represents a specimen, No. 826, which

has been bent upward at the cephalic end. The fang is entire, but the other teeth have been broken off. The bases of four of them remain. The fine teeth are very short in this specimen.

	mm.
Length of tooth border	35
Greatest depth of the dentary	8
Length of the dentary fang	4.5
Length of the fine teeth	0.9

Figure 7, plate XV, represents specimen No. 323, the largest specimen of this species in the collection. Only the cephalic end of the dentary is present. Four small teeth occur on the external alveolar border in front of the fang. One is broken, but the remaining three are one-third as long as the fang.

	mm.
Length of the dentary fang	9
Length of the teeth in front of the fang	2

Figure 8, plate XV, shows specimen No. 838, a much smaller palatine, with its fang. The angle that this palatine makes with its fang is 140°. A portion of the pterygoid adheres to the caudal end of the palatine.

	mm.
Length of the palatine fang	4

Figure 9, plate XV, represents a palatine complete with its fang, which very probably belongs to this species. It is No. 828. The palatine is not quite half as long as the fang. The angle that the palatine makes with the fang is about 150.° It is probably from a young specimen, since the length of the palatine would indicate that the fang had not been replaced many times.

	mm
Length of the fang	6

There is much variation in the size of this species. This is well shown by a comparison of the lengths of the dentary fangs. The lengths of several of these fangs are as follows:

	mm
Fig. 4, specimen No. 321	2.5
Fig. 3, specimen No. 824	3
Fig. 6, specimen No. 826	4.5
Fig. 1, specimen No. 829	6
Fig. 5, specimen No. 825	7
Fig. 7, specimen No. 323	9

The generic names designated in the bibliography are now generally considered to be synonymous. They are thus given

although complete verification by the author has been impossible because of the inaccessibility of some of the publications and the meager accounts furnished in others.

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THE

KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 3—January, 1913.

(Whole Series, Vol. XVII, No. 3.)

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PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 3] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No. 3

A PLEISTOCENE MOLLUSCAN FAUNA FROM PHILLIPS COUNTY, KANSAS.

BY G. DALLAS HANNA AND EDWARD C. JOHNSTON.

Plate XVIII.

(Contribution from the Zoological Laboratory, No. 202.)

RECENTLY, in looking over some stored fossils, Mr. H. T. Martin, of the University of Kansas Museum, found along with some vertebrates a small series of fossil shells from Long Island, Kan. The vertebrates were collected by E. P. West, and it is probable that the shells were also collected by him, although there was no name on the label. An opportunity to study this material was given the writers by Prof. C. E. McClung, to whom we are duly thankful. In regard to the topography and history of the region Mr. Martin was able to give much interesting information.

A hurried examination of this material showed that it belonged to a fauna essentially different from that which lives at the present time not far distant from Long Island, in localities where collecting has been done. Most of the Long Island species were not represented in our collections from points nearer than northern United States.

In order to obtain a more complete series of the shells and data in regard to the horizon in which they occur, a trip was made to the region in October, 1910. We started at Norton, twenty-five miles southwest of Long Island, and descended Prairie Dog creek to that point. Recent shells were collected along the route and fossils at many railroad cuts and exposures along the creek. The white Miocene rocks capping the hills

Received for publication June 20, 1912.

were also examined for shells. At Long Island, along the creek bank, is an especially good exposure of Pleistocene. The Miocene of the locality is famous for its vertebrates.

As three faunas are represented in the one locality, it is well to consider them separately.

RECENT MOLLUSKS OF THE REGION.

The living mollusks of the locality belong to that fauna designated by the U. S. Biological Survey* as Upper Sonoran. As to shells, this fauna is characterized not so much by the development of peculiar species or subspecies as by the absence of a very large proportion of species belonging to the Carolinian zone to the eastward. West of the Mississippi river these species gradually fade out, some species and groups not extending as far as others. If the westward limit of the most characteristic of Carolinian shells, the large toothless *Polygyras,* be used for the boundary between the Carolinian and Upper Sonoran zones, we find the Carolinian extending only as far west as the vicinity of Topeka, Kan. This is somewhat to the eastward of the boundary as plotted on the faunal map.

Phillips county, lying far west of this, is wholly beyond the influence of the Carolinian zone, and its fauna is typically Upper Sonoran. This zone and fauna extends northward almost to the northern boundary of the United States, beyond which lies the transition realm; between these two there is little intermingling, the latter being characterized by a large number of specialized species.

Mollusks are not abundant in the Upper Sonoran, and belong chiefly to the groups of small species; the *Succineas* are the largest of the region. Each group, as a rule, has a different habitat, but a favorable locality for many species is along the base of the crumbling Miocene ledges at the tops of the hills. Trees and shade are confined to the very banks of the creeks, but many shells can be found on the bare prairies about the bases of grasses, brush and cactus, where this is present.

The streams are almost destitute of shells. This is doubtless due to their sandy character and the fact that they go dry in some seasons. Occasional deep holes in Prairie Dog creek are, however, favorable for bivalves, and four species were collected.

* U. S. Biological Survey, Fourth Provisional Zone Map of North America; Merriam, Bailey, Nelson, Preble, 1910

LIST OF RECENT SPECIES.

Sphærium sp?
Lampsilis parva Lea.
Quadrula lachrymosa Barnes.
Anodonta sp?
Physa anatina Lea.
Planorbis bicarinatus Say.
Zonitoides miniscula Binney.
Zonitoides singleyana Pilsbry.
Vallonia gracilicostata Reinhardt.
Vallonia parvula? Sterki.
Pupoides marginatus Say.
Bifidaria procera cristata P. & V.
Bifidaria armifera Say.
Bifidaria holzingeri? Sterki.
Bifidaria tappaniana C. B. Adams.
Succinea grosvenori Lea.

MIOCENE OF THE REGION.

On Prairie Dog creek, between Norton and Long Island, there are many fine exposures of what is generally known as Loop Fork. According to Matthew* this belongs in the upper Miocene series, and the short-legged rhinoceroses are the most abundant fossils.

The upper layer of this formation is the most conspicuous, as a hard layer of porous white limestone caps most of the hills along the route. This layer weathers poorly, and in the majority of places exposes a perpendicular section on the hillside. Beneath this there are many layers of sand and other rocks, which are well exposed in a deep ravine south of Long Island about five miles. At Almena a beautiful yellow sand is exposed in great quantities. The layer is about forty feet thick, and two mastodon skeletons are exposed.

All of the strata are fossil-bearing, rhinoceroses being found most abundantly, but many other groups are common. *Euphorbia* seeds were found in all the layers examined.

The formation is doubtless of Miocene lake origin, and may contain fossil land shells in some places, but none were found by us. These are occasionally found in the John Day region in Oregon and the White river Miocene of Colorado, from which it may be inferred that they exist sparingly in the Loop Fork. Those land shells found in the former two divisions of the Miocene belong to a warmth-loving fauna, and as they are not greatly different in general character from species inhabiting the Carolinian zone at the present time, it may be supposed that the climate of the Miocene period was not essentially different from that of the region at the present time.

In later times there has been a gradual increase in cold,

* W. D. Matthew, Bull. Am. Mus. Nat. Hist. XII, p. 65.

which reached its climax in the southern extension of the glaciers to northern Kansas, and still later, of course, the former conditions have returned.

PLEISTOCENE OF THE REGION.

To the Pleistocene we refer a large number of shell-bearing deposits found all the way along Prairie Dog creek between Norton and the Republican river. Some of the deposits are of uniform consistency throughout and seem to be of aërial origin, while others are distinctly laminated and contain some gravel and nodules of white disintegrated Miocene rock, indicating that they are of fresh-water origin.

One mile east of Long Island the bed of the creek is formed of a comparatively tough green marl, which contains an abundance of fossil shells. The creek has cut down through this marl, which is not all of the same consistency, for a distance of thirty feet. Shells are scattered all through this, except in the upper ten feet. The marl seems unmistakably of fresh-water origin, and most of the fresh-water species were found in it.

West of Long Island about three miles there is an exposure of about twenty feet of sand lying on top of Miocene shales, which form the bed of the creek. The deposit contains an abundance of shells, but owing to the few *Lymnæas* found, and structure of the whole, it is supposed to be of wind origin. This deposit was the only one seen which had the columnar structure of the Mississippi and Missouri river valleys. Along the latter streams the loess exists as a chain of hills along the banks of the streams, but if such a condition ever existed along Prairie Dog creek, erosion has eliminated all traces of it. The deposit under consideration could have been drifted by the wind, owing to the position of the surrounding Miocene hills.

The species found in the deposits, which seem to be of wind formation, are, with the exception of most of the fresh-water ones, the same as those found in the marl east of Long Island. The latter is of undoubted water origin, and all seem to be of contemporaneous deposition.

In the loess of the Mississippi and Missouri river banks there are an abundance of fossils belonging to species, with one or two exceptions, exactly the same as inhabit the region at the present time. They have lived in the locality ever

since the beginning of the formation of the beds, because they are found from the bottom up. But in the Prairie Dog creek beds there are found no less than twenty-six species which do not live to-day within several hundred miles of the formation. They are as follows:

Psidium sp?
Psidium sp.
Sphærium sp.
Ancylus parallelus?
Lymnæa parva.
Lymnæa obrussa?
Lymnæa humilis.
Lymnæa desidiosa.
Planorbis trivolvis.
Planorbis truncatus.
Planorbis deflectus.
Zonitoides arborea.
Euconulus fulvus.
Vitrea hammonis.
Pyramidula cronkhitei cockerelli.
Punctum pygmæum.
Sphyradium alticola.
Sphyradium hasta.
Pupilla muscorum.
Bifidaria mcclungi.
Vertigo modesta.
Vertigo gouldi.
Vertigo martini.
Succinea avara.
Succinea sillimani.
Carychium exile canadense.

Great changes in climate must have taken place in order that this number of species should completely die out of the region. Many of these twenty-six species never could have lived on a barren, sandy, treeless plain. The entire surroundings must have been vastly different in regard to water and vegetation for the existence of such moisture-loving forms as *Vertigo, Carychium, Euconulus,* etc.

Of these twenty-six species fully 50 per cent are associated at the present time with a cold climate. They live in Canada and the northern parts of the United States and among the peaks of the Rockies. The natural conclusion to be drawn from this is that the Prairie Dog creek fauna lived during a cold epoch.

It may be stated that the fauna is as different from the loess fauna as from the recent fauna in the loess region.

It is generally admitted that the loess deposits are post-glacial. Their existence on the top of glacial drift indicates this, as also does the presence of a uniform fauna from bottom to top. If the deposits were formed during and after glacial time the bottoms would contain a different fauna from the top. The presence of the northern fauna in the Prairie Dog creek beds is good evidence that it is older than that of the loess, and that it lived during an invasion of cold from the north.

The Glacial Epoch, during which there may have been as many as five ice sheets in some localities, is the only descent

of cold known to have occurred in later geological times. The glaciers extended as far south as Lawrence, Kan., but there seems to be no evidence of the ice in Phillips county, which is in a somewhat more northern latitude. However, the existence of glaciers in or near the eastern part of the state would have a great effect upon the animals and plants of the western part of the state, because temperature conditions would be essentially the same. From this it seems that the Prairie Dog creek beds were deposited immediately preceding and during the Glacial Epoch. They will be placed provisionally in the upper Pleistocene.

The existence of this fauna so far southward is evidence that goes to prove the gradual beginning and long duration of the Glacial Epoch. Land snails are among the slowest of animals to be geographically distributed, and the time taken for this Canadian fauna to migrate several hundred miles to the southward must have been of considerable magnitude. It existed in the Long Island region during the deposition of at least thirty feet of sand, and died out when conditions changed and the glaciers disappeared. A few remnants of the fauna exist at the present time in eastern Kansas as *Pupilla muscorum* and *Euconulus fulvus.*

As most of the fauna is found in the Miocene of the locality, it seems that the descent of cold had not started at the close of that period.

There are some shells which we did not find in the deposits, but which would naturally be supposed to occur. On subsequent search a *Vitrina,* for instance, may be found. None of the Helicidæ were found as fossils. None of the family lives in the region at the present time, although *Polygyra fraterna* lives in Jewell county to the eastward.

ANNOTATED LIST OF PLEISTOCENE SPECIES.

1. **Psidium** (sp?)

2. **Psidium** (sp?)

3. **Sphærium** (sp?)

Three species of small bivalves were found. The first two are rare and the latter most common.

4. **Ancylus parallelus?** Say.

A single specimen found at Long Island probably belongs to this species.

5. **Lymnæa obrussa?** Say.

Several specimens were found which more rearly resemble this than any other species in our collections. The aperture, however, is shorter and broader.

6. **Lymnæa parva** Lea.

Fifteen specimens from the Long Island marl.

7. **Lymnæa desidiosa** Say.

8. **Lymnæa humilis** Say.

The latter two species were quite common in the marl beds.

9. **Physa anatina** Lea.

This is a smooth *Physa*, agreeing with specimens of *anatina* from eastern Kansas identified by Mr. Bryant Walker.

10. **Planorbis trivolvis** Say.

This widely distributed species does not live in Prairie Dog creek at the present time. Several fossils were found.

11. **Planorbis truncatus** Say.

One specimen found at Long Island.

12. **Planorbis bicarinatus** Say.

Numerous specimens were found in the green marl near Long Island, and the species lives in Prairie Dog creek at the present time.

13. **Planorbis deflectus** Say.

Two specimens found.

14. **Zonitoides miniscula** Binney.

A few fossils found in a railroad cut just east of Norton.

15. **Zonitoides miniscula alachuana** Dall.

Fossils were found in two or three deposits. The shells are much smoother than recent specimens of either *alachuana* or typical *miniscula*.

16. **Zonitoides arborea** Say.

Several fossils found. The last whorl is rather more flattened than in recent specimens, and all trace of the brownish color was lost. In Kansas, this species does not live west of Topeka at the present time.

17. **Euconulus fulvis*** Muller.

Quite common as a fossil, and typical. Fine spiral sculpture over the entire shell, and the whorls are almost six.

18. **Vitrea hammonis** Müller.

Several fossils were found in the different beds. The last whorl in-

* See Pilsbry and Ferris. Proc. Acad. Nat. Sci Phila., 1910, p. 133.

creases rather more rapidly than in recent specimens. Lawrence is the farthest west the form lives in Kansas at the present time.

19. **Pyramidula cronkhitei cockerelli** Pilsbry.

One of the most abundant fossils. It can be collected by thousands. It lives at the present time no closer than the mountains of central Colorado.

20. **Helicodiscus parallelus** Say.

Common in all the beds.

21. **Punctum pygmæum** Draparnaud.

Rare; only a few specimens could be found, but owing to its small size it may have been overlooked.

22. **Vallonia gracilicostata** Reinhardt.

An abundant fossil in all the deposits examined.

23. **Vallonia parvula** Sterki.

Five specimens were found in a deposit of loam, near the surface, four miles east of Norton. This probably does not extend down into the Pleistocene formation, and it is doubtful if it lives in the region at the present time.

24. **Cochlicopa lubrica** Müller.

A common shell in the Pleistocene beds. In Kansas this shell is rare and does not live in the region at the present time. Hutchinson is apparently the western limit.

25. **Sphyradium alticola** Ingersoll.

This form is specifically distinct from *Sphyradium edentula.* The shell is much longer and the basal whorl is always expanded beyond the others. The whorls in *edentula* are five to six, while in *alticola* there are seven to eight.

Measurements of S. alticola.

Alt.	Diam.
3.33	1.74
2.81	1.11
2.70	1.14

The species is very common in the formation, and is said to live in the mountains of Colorado at the present time.

26. **Sphyradium hasta** Hanna & Johnston. Pl., fig. 5.

This form is rare in the deposits. The type is from the green marl formation and is deposited in the U. S. National Museum.

27. **Pupilla muscorum** Linnæus.

This is the most abundant fossil of the series. A very large number were collected, in which considerable variation is found. The edentulous form is the variety which occurs in greatest numbers, with those that have a single nodule-like tooth on the parietal wall not far behind. Two

forms of a bidentate *muscorum;* one, the most common of the two, has one tooth on the parietal wall and one on the columella; in the other the columella tooth is absent and is replaced by one on the palatal wall. The latter form is what is known as *bigranata,* but only a few specimens were found, while the former is quite common. All four of these forms intergrade to a remarkable degree, and the fact that all are found in the one locality makes the recognition of subspecies not feasible.

28. **Pupilla blandi** Morse.

This form has three teeth, and the shell is smaller and more cylindrical than *P. muscorum.* A few fossils were found. The species now lives in the mountains of Colorado.

29. **Bifidaria mcclungi** new species. Pl., figs. 1, 2.

Shell light brownish-yellow; composed of five to five and a half whorls. Lines of growth faint and apex smooth. Whorls well rounded and suture very deep. Umbilicus barely perforate. Peristome very broadly reflected and greatly thickened by white callous on the inner margin and the face; ends connected across the body whorl by a heavy callous. Teeth six. One parietal, large, heavy, and strongly bifid. The angular or outer portion from the columella is connected to the peristome by its upper extremity. It descends in a gentle curve and meets the parietal portion near the upper extremity of the latter. Both portions of the tooth are often divided transversely, as shown in figure 2. One columellar, large and heavy, and with a small, pointed subcolumellar. This is sometimes partly united to the upper tooth, in which case it is bifid, as shown in figure 1. One basal, small and conical. Lower palatal high and lamellar, set oblique to the aperture, and more deeply seated than the other teeth. Upper palatal small and conical. Immediately above the upper palatal tooth there is a nodule-like thickening of the margin of the lip. A low callous ridge on the outside of the shell just back of the reflected lip.

Measurements.

Alt.	Diam.
2.03	.96
2.07	1.00
2.44	1.00

Types in the U. S. National Museum from the Pleistocene of Phillips county, Kansas.

This species differs from any other of the *procera* group of *Bifidaria* in having the compound tooth on the columella. The very greatly thickened and expanded lip is also characteristic.

30. **Bifidaria holzingeri** Sterki.

Three specimens were found associated with *Vallonia parvula* in loam two feet below the surface. It probably does not extend down into the Pleistocene deposits, and it is doubtful if it lives in the region at the present time.

31. **Bifidaria armifera** Say.

Typical *armifera* is found as a fossil, and lives abundantly in the region at the present time.

32. **Vertigo modesta** Say.

Very common in all of the deposits. It is not found living in the state. The second parietal tooth is never developed in these specimens, and the shells are smaller and more cylindrical than the typical *modesta.*

Measurements.

Alt.	Diam.
2.02	1.07
1.97	1.00
1.80	1.16

33. **Vertigo gouldi** Binney.

Very common in the deposits. The columellar tooth is much heavier and the lower palatal more deeply seated than in the typical form, but we do not hesitate to refer the shells to this species.

34. **Vertigo martini** new species. Pl., fig. 3.

Shell light brown; ovate in outline. Lines of growth faint and oblique. Whorls, four and a half, well rounded, and the sutures well impressed. Apex smooth and white and obtusely pointed. Peristome thin and sharp, slightly expanded, and the ends connected across the body whorl by a thin deposit of callous; almost no indentation in the upper palatal wall. Aperture semicircular and with six teeth. Two on the parietal wall, both of which are lamellar in shape, and the angle tooth is the smaller of the two. One columellar in the center of that wall of the aperture. This tooth is bifid, that portion toward the apex of the shell being the larger. One basal tooth, small and nodule-like. Two palatals, both of which are lamellar in shape, the lower one of which is the larger. Variation in a large series of this species is slight.

Measurements.

Alt.	Diam.
1.77	1.04
1.74	1.04
1.63	1.04
1.55	1.04

Type specimens in the U. S. National Museum from the Pleistocene of Phillips county, Kansas.

This and *Vertigo ovata* are the only species found in this part of the country with two teeth on the parietal wall. The latter species, however, is much the larger and more ovate in outline. The size of *martini* is about the same as that of gouldi from the same deposits, but that species is more cylindrical and the angle tooth is never developed. The size is somewhat less than that of *Vertigo tridentata,* a recent shell of eastern Kansas in which the basal tooth is absent.

35. **Succinea grosvenori** Lea.

An abundant fossil, and found living continuously along the bluffs of Miocene exposures. Some of the shells are very large, reaching 19 mm. in altitude and 13 mm. in diameter. These are some of the heaviest *Succineas* in the country. The heavier shells intergrade perfectly with the smaller, lighter ones, so it is supposed that the larger ones are only older individuals.

36. **Succinea avara** Say.

Very common in all the beds. The shells are normal and do not live nearer than central Colorado or eastern Kansas at the present time.

37. **Succinea sillimani** Bland.

Rare; only about a dozen specimens found.

38. **Carychium exile canadense** Clapp.

Naut. XIX, p. 138; pl. VII, figs. 1, 2, 6, 7; Apr., 1906.

This form is rare in the deposits. Only a few were found, and they agree with the form above except in being more slender and the internal basal ridge is more strongly developed than the figures show. The striæ are not numerous, but are high and coarse and resemble ribs on the body whorl, but they are not uniform and regular.

THE

KANSAS UNIVERSITY SCIENCE BULLETIN

Vol. VII, No. 4 — January, 1913.

(Whole Series, Vol. XVII, No. 4.)

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SOME TREMATODES OF KANSAS SNAKES.....*H. E. Crow.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 4] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No. 4

SOME TREMATODES OF KANSAS SNAKES.

BY H. E. CROW.

Plate XIX.

(Contribution from the Zoölogical Laboratory, No. 203.)

Submitted in partial fulfillment of the requirements for the degree of Master of Arts.

INTRODUCTION.

DURING the summer of 1911, as a member of the State Biological Survey, I had the opportunity of examining a large number of snakes, frogs, turtles, fish, etc., for parasitic worms. All of the forms were more or less infested, so quite a number of worms were procured. Some of these have been studied with the object of determining the distinctive characteristics and the placing of them in their respective systematic position. Lack of time has prevented the presentation of more than three species in this paper. All three worms were found in snakes, and are new to science.

The worms were placed directly into 70 per cent alcohol. This causes a very slight shrinkage, but they stain well in any stain and clear perfectly. Borax carmine was used with best results in staining both whole mounts and sections. Hæmalum, picrocarmine and Wright's blood stain were also used with varying results. Specimens mounted in glycerine showed the various ducts very distinctly.

These worms differ in some respects, but they are so closely related that there can be no doubt they belong to the same genus. While they all differ in some minor points from the type specimen, they surely belong to the genus *Renifer*, characterized by Pratt as follows:

Small distomes, with body more or less elliptical and covered with minute spines or scales. Mouth subterminal; acetabulum somewhat

Received for publication September 23, 1912.

larger than oral sucker, and in anterior half of body. Pharynx present; œsophagus short or wanting; intestinal cæca reach about to the middle of the body or a little past it. Excretory vesicle very large, Y-shaped, extending into the forward end of body. Genital pore in front of the acetabulum often to the left or the right of the median line. Testes two, usually lobate, in the same transverse plane near the middle of the body; cirrus-sac often long, containing a *vesicula seminalis*, and in front of the testes; yolk glands lateral in position and occupying the middle third of the body; Laurer's canal present; uterus composed of a descending and an ascending limb, and extending to the hinder end of the body; *receptaculum seminis* absent or minute. Parasite in the mouth or air-passages of snakes and in the intestines of turtles.

TYPE: *Renifer ellipticus* Pratt. Additional species: *R. elongatus* Pratt, *R. variabilis* (Leidy) Luhe, *R. solitarius* Looss, *R. zschokkei* Volz, *R. acetabularis* Crow, *R. megasorchis* Crow, *R. kansensis* Crow.

Renifer megasorchis sp. nov.

This worm was found attached to the wall of the uterus of a large *Natrix rhombifera*, a common nonpoisonous snake. It is a comparatively large worm, pinkish in color and slightly lanceolate in form. The body is widest just posterior of the middle and tapers to blunt points at both ends, but more rapidly to the posterior end. The ventral surface is flat or slightly elliptical, the dorsal surface cylindrical. In length the body measures 5 millimeters; the width at the widest point, 1.72 millimeters; thickness, .92 millimeter. The body is covered with short spines arranged in transverse rows, the spines in each row alternating with the row proximal to it. The spines are slightly longer and less numerous towards the posterior end of the body. The two suckers are sessile. The oral sucker is subterminal and measures .46 millimeter in length and .43 millimeter in width. The acetabulum is slightly larger than the oral sucker and is .48 millimeter in diameter. It is situated in the median line in the second fourth of the ventral surface.

The digestive tract consists of pharynx, œsophagus and intestinal cæca. The pharynx measures .247 millimeter in length and .226 millimeter in diameter. It is a very muscular organ, the radially disposed fibers constituting the principal part, with a very thin layer of circular fibers on the outside. The œsophagus is a small, round, straight tube, .06 millimeter in diameter and .32 millimeter long. Both the pharynx and

œsophagus are surrounded by glands. The intestinal cæca are simple tubes, without lateral projections, extending to the beginning of the last fourth of the body. The right tube is slightly longer than the left. They lie in about the same plane and follow the general body outline. They are generally round, with an average diameter of .09 millimeter.

The excretory vesicle is a Y-shaped tube, with very thin walls—so thin that it can be traced only in sections. From the small opening at the extreme posterior end of the body the tube proceeds forward, gradually growing larger till it reaches the anterior end of the testes, where it divides, each branch passing forward just inside its respective intestinal cæcum to about the posterior edge of the acetabulum. From the bifurcation the crura grow rapidly smaller and terminate as blunt points. At the widest point the vesicle measures .91 millimeter in width, while the dorsoventral diameter at the same point is only .04 millimeter. At the neck, or just inside the pore, the diameter of the thin cylindrical tube is .036 millimeter. The neck is surrounded by glands which define its limits clearly. The position of the excretory vesicle is dorsal to all the other organs except the yolk ducts.

The testes are two very large oval bodies, situated in the third fourth of the body, and nearly filling the body cavity at its widest point. The left testis is slightly in advance of the right one. They are the same size and measure .98 millimeter in length, .61 millimeter in width and .77 millimeter in thickness. The *vasa efferentia* arise from the anterior surface of each testis and pass forward dorsal to the uterus and to the left of the ovary to the posterior end of the cirrus-sac, where they join to form the *vesicula seminalis*. The *vasa* are very small tubes, .0098 millimeter in diameter, and can be seen only in sections. The *vesicula seminalis* is entirely within the cirrus-sac, nearly filling it at the posterior end. In its course it bends upon itself and is continued as a short *pars prostatica*. The cirrus which follows is a large, long organ, and extends through the cirrus-sac and outside the body wall a distance of .528 millimeter. Its length within the cirrus-sac is about equal to that outside the body. It measures .092 millimeter in diameter. The cirrus-sac is a large organ enclosing the cirrus, *vesicula seminalis*, *pars prostatica* and prostate glands. It measures .363 millimeter in diameter at the widest point—

the posterior end—and tapers gradually to a diameter of about .12 millimeter. The prostate glands occupy all the cavity of the cirrus-sac not taken up by the other enclosed organs.

The ovary is situated a little to the right of the median line against the dorsal body wall, just posterior the cirrus-sac. It is oval in shape, measuring .415 millimeter in length and .332 millimeter in diameter. The oviduct arises from the left dorsal surface of the ovary as a small short tube, and passes at once into the small oötype, which is surrounded by the shell gland. There is no *receptaculum seminis.* From the oötype a slender Laurer's canal passes slightly posteriorly to the dorsal body wall, having its external opening in the median line. The yolk glands consist of about fifty rounded bodies on each side of the body, mainly outside of the intestinal cæca, extending over about the middle third of the body. A main duct from the anterior and one from the posterior parts of each gland join to form the two transverse ducts which join in the median line to form the common yolk duct, which enters the oötype on the posterior side. On the ventral side of the oötype the uterus arises as a small tube, which soon turns posteriorly, and as it passes toward the hinder end of the body grows larger and pursues a rather serpentine course. Near the posterior end of the body the uterus turns back and passes anteriorly, dorsal to the descending limb, to a point to the left of the middle of the cirrus-sac, where it enters the metraterm. The ascending limb becomes very voluminous as it passes forward and occupies a large part of the body cavity. In younger specimens the limbs of the uterus are small and of the same diameter, but in the adult the ascending limb is many times larger, and, being filled with a dense mass of eggs, often hides many of the other structures. The egg measures .031 millimeter in length and .016 millimeter in width. The metraterm lies in the same plane as the cirrus-sac and to its left. It passes in a curved course to the genital pore, which lies in the median ventral line a short distance anterior to the acetabulum, immediately anterior to the cirrus. It is a comparatively thick-walled structure, with many folds in its inner wall. In diameter it measures .146 millimeter, and is five or six times as long.

Specific diagnosis of *Renifer megasorchis:* Slightly lanceolate, semicylindrical worms, 5 millimeters long, 1.72 milli-

meters wide, and .92 millimeter thick. Body covered with short spines. Oral sucker subterminal, .46 millimeter in length, .43 millimeter in width. Acetabulum sessile, .48 millimeter in diameter. Pharynx moderate; œsophagus moderately long; intestinal cæca simple, reaching to third fourth of body. Excretory vesicle Y-shaped; long, voluminous median portion; crura short and extend very little past the middle of the body. Testes two very large oval bodies; the left slightly anterior to right; behind ovary and acetabulum. Ovary immediately behind acetabulum; large and oval in shape. Uterus very voluminous; metraterm long and muscular, genital pore in front of acetabulum in median line. Egg .031 millimeter long and .016 millimeter wide. Laurer's canal small; no *receptaculum seminis*. Cirrus-sac large; cirrus extends .528 millimeter beyond body wall. Yolk glands lateral, occupying middle third of body. Parasite in the uterus of *Natrix rhombifera*.

Renifer acetabularis sp. nov.

This worm was found in the mouth of *Natrix rhombifera*. It is elliptical in outline, slightly concave on the ventral side, while the dorsal side is cylindrical, a cross section thus being almost a half of a sphere. In all of the specimens there was a characteristic downward bend from the acetabulum forward. In length the worm measured 1.56 millimeters, and .62 millimeter in width at the middle of the body. The thickness at this place was .151 millimeter. The body is covered with short, stiff spines. The suckers are very large and sessile. The oral sucker is subterminal in position and measures .286 millimeter in length and .241 millimeter in width. The mouth cavity occupies a little more than half of the space inclosed by the sucker. The acetabulum, a very large, round, sessile organ, is situated on the mid-ventral surface. It measures .453 millimeter in diameter.

The digestive system is made up of mouth, pharynx and two intestinal cæca. The mouth is a funnel-shaped structure, opening from the oral sucker directly into the pharynx. The pharynx measures .081 millimeter in length and .074 millimeter in diameter. The opening through the pharynx is nearly one-third of the whole diameter of the organ. The intestinal cæca branch off directly from the posterior end of the pharynx and

extend to the posterior end of the middle third of the body, almost to the testes. They are very thin-walled tubes, with many shallow outpouchings. They are about the same diameter at all points, except at the very end, where there is a slight enlargement The average diameter is .064 millimeter.

The excretory vesicle is a Y-shaped, very voluminous organ. Beginning with a very small pore at the posterior end of the body, the vesicle extends anteriorly, suddenly becoming much larger and filling the greater part of the body cavity for a short distance. It becomes narrow to pass between the testes, but widens again, and soon bifurcates, each branch passing forward just within the intestinal cæca, and, gradually growing smaller, ends near the bifurcation of the intestinal cæca. The course is dorsal to all other organs, through the very tips of the crura pass partially ventral to the intestinal cæca. The walls are very thin, and, except a short distance at the posterior end, can be traced only in sections.

The testes are two partially lobate bodies, situated just posterior to the middle third of the body in the same transverse plane. They differ somewhat in shape, but they are approximately the same size. The shape is variable in different individuals. The average size is .144 millimeter in length and .126 millimeter in width. The *vasa efferentia* pass from the anterior end forward and join the *vesicula seminalis* separately. The last-named organ, together with the *pars prostatica* and cirrus, are enclosed in the long cirrus-sac, which begins just to the left of the median line anterior to the acetabulum and passes in a curving course to the genital pore on the left ventral surface, near the oral sucker. The cirrus-sac is .456 millimeter in length and .052 millimeter in diameter at the largest point. The *vesicula seminalis* bends on itself in the posterior end of the cirrus-sac. It is followed by a very short *pars prostatica,* which in turn is followed by the cirrus. The cirrus is small and weak.

The ovary is situated just posterior to the middle of the body and to the right of the median line. It is nearly spherical, measuring .105 millimeter long and .098 millimeter in diameter. Slightly dorsal and to the left is the shell gland surrounding the small oötype. The short oviduct joins the oötype on the anterior and the yolk duct on the posterior side. From the ventral side of the oötype the uterus proceeds pos-

teriorly in a serpentine course to near the hinder end of the body. Here it turns and passes ventral to the descending limb to the branching of the intestinal cæca, and then diagonally to the genital pore. No metraterm is developed. The eggs with which the uterus is filled are .032 millimeter long and .016 millimeter wide.

The yolk glands are a number of rounded or elongated bodies, averaging about .04 millimeter in diameter, lying in two groups on each side of the body. One group on each side lies lateral and mainly anterior to the respective testis. The other group is situated at the anterior edge of the acetabulum, mainly ventral and lateral to the intestinal cæca. There are about fifteen bodies in each group. A duct from the anterior group meets one from the posterior group and form the transverse duct on each side. These join near the ovary, and form the common yolk duct which enters the oötype.

Specific diagnosis of *Renifer acetabularis:* Elliptical, semicylindrical worms, 1.56 millimeters long and .62 millimeter wide; body covered with short spines. Suckers very large and sessile; oral sucker subterminal, .286 millimeter in length and .241 millimeter in width; acetabulum circular, .453 millimeter in diameter, and in center of ventral surface. No œsophagus; pharynx and intestinal cæca moderate; latter with many shallow outpouchings; reaches to posterior end of middle third of body. Excretory vesicle very large, Y-shaped; crura extend almost to pharynx. Testes two small, slightly lobate bodies in posterior third of body, in same transverse plane, behind ovary and acetabulum. Ovary immediately dorsal to acetabulum, nearly spherical, and moderate in size. Uterus voluminous, extends to near posterior end of body. Genital pore to left of oral sucker on ventral surface. Egg .0321 millimeter long, .0162 millimeter wide. No oötype, Laurer's canal or *receptaculum seminis* seen. Cirrus-sac large, encloses cirrus, *pars prostatica* and *vesicular seminalis;* yolk glands in two groups on both sides of body. Parasite in mouth of *Natrix rhombifera.*

Renifer kansensis sp. nov.

Several specimens of this worm were found attached in the mouth of *Ancistrodon contortrix*—the copperhead snake. It

is nearly elliptical in outline; the dorsal body is oval, the ventral flat. The body is covered with short, stiff spines, arranged somewhat regularly. The worm is 4.56 millimeters long, 1.18 millimeters wide, and .387 millimeter thick. The suckers are nearly the same size and sessile. The oral sucker is subterminal and measures .392 millimeter in length and .356 in width; the opening has a diameter of .127 millimeter. The round acetabulum is situated in the second fourth of the body. It measures .453 millimeter in diameter. The genital pore is on the ventral surface, very close to the left edge of the body, in same transverse plane as the posterior end of the pharynx. The excretory pore is at the posterior end of the body.

The digestive system is made up of the pharynx, œsophagus and intestinal cæca. The pharynx is spherical, .21 millimeter in diameter. An œsophagus .36 millimeter in length follows. It increases slightly in diameter as it passes posteriorly, and divides into the two intestinal cæca. These are simple tubes, which reach to the middle of the body. They have a diameter of .07 millimeter at the widest point. The left one is slightly longer than the right.

The excretory vesicle is long and voluminous. It has the shape of the letter Y. The median portion extends forward to the anterior end of the testes, and the crura almost reach the branching of the digestive tract. The median portion has a great number of lateral, irregular outpouchings, which posterior to the uterus nearly fill the body cavity. The walls are very thin and, even in sections, are difficult to trace. The crura, however, are more definite tubes, which are more regular in their course. They lie mainly dorsal to the intestinal cæca, and for the most part parallel. They are generally round and about half the diameter of the intestinal cæca. The position of the excretory vesicle is dorsal to the other organs.

The testes are two small lobate bodies, situated slightly posterior to the middle of the body. They are generally round, though very irregular in shape, averaging .275 millimeter in diameter. The very small *vasa efferentia* pass from the anterior ends of the testes forward to a point a short distance anterior to the acetabulum, where they enter the *vesicula seminalis*. This organ, the *pars prostatica* and cirrus are enclosed in the cirrus-sac. The cirrus-sac is very long, measuring about .50 millimeter in length. It lies ventral to the left intestinal

cæca, and passes diagonally to the gential pore at the left side of the body. The *vesicula seminalis* makes a spiral bend in the posterior end of the cirrus-sac and nearly fills that part. A short *pars prostatica* follows, then the cirrus continues through the cirrus-sac. The end of the cirrus protrudes a short distance.

The ovary is a small, round organ, .22 millimeter in diameter, which lies partly behind and dorsal to the acetabulum. It is dorsal in position and just to the right of the median line. The short oviduct passes toward the median line and enters the oötype, which lies in the midst of the shell gland. The shell gland lies to the left and slightly dorsal to the ovary. No *receptaculum seminis* or Laurer's canal are present. The yolk glands consist of about fifty small, irregular-shaped bodies on each side of the body. They are lateral and mainly ventral to the intestinal cæca, and extend over a little more than the middle third of the body. Several small ducts from each gland join to form the lateral duct, which proceeds toward the median line, where it meets its fellow. The common yolk duct thus formed passes anterior a very short distance and joins the oötype on the posterior side. The uterus is a very voluminous organ. It consists of a descending and an ascending limb, both of which have a serpentine course. The descending limb passes from the oötype along the median line to near the hinder end of the body. The ascending limb proceeds forward between the testes to the branching of the intestinal cæca, where it passes ventral to the left one to the genital pore at the left side of the body. There is no specialized metraterm. The ascending limb is several times larger than the descending, and fills about half of the body cavity behind the testes. The uterus is filled with eggs, which measure .045 millimeter by .019 millimeter.

The following is a specific diagnosis of *Renifer kansensis:* Small, elongated, semicylindrical worms, 4.56 millimeters long, 1.18 millimeters wide, and .387 millimeter thick. Body covered with short spines. Oral sucker subterminal, .392 millimeter long and .356 millimeter wide. Acetabulum sessile, .453 millimeter in diameter. The pharynx is small; œsophagus moderate; intestinal cæca simple tubes, which reach to middle of body. Excretory vesicle voluminous, Y-shaped, with many outpouchings in median portion; crura and median portion

long. Tests two, slightly posterior to middle of body, behind ovary and acetabulum. Ovary partly behind and partly to the right of acetabulum. No Laurer's canal or *receptaculum seminis* present. Yolk glands lateral, extending over middle third of body. Genital pore near left edge of body, near oral sucker. Cirrus-sac long and lying diagonally across body. Uterus very voluminous. Egg measures .045 millimeter by .019 millimeter. Parasite in the mouth of *Ancistrodon contortrix*.

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THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 5—June, 1913.

(Whole Series, Vol. XVII, No. 5.)

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THE BIRDS OF KANSAS.....*C. D. Bunker.*

PUBLISHED BY THE UNIVERSITY

LAWRENCE, KAN.

Entered at the post office in Lawrence as second class matter

1 5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 5] JUNE, 1913. [WHOLE SERIES VOL. XVII, No 5

THE BIRDS OF KANSAS.

C. D. BUNKER.

(Contributions from the Zoological Laboratory, No. 204.)

THE geographical position of Kansas makes the study of its bird life particularly interesting. The state has a broad range of varying faunal areas. It is divided north and south by the meeting of the Upper and Lower Austral Zones, near the southern boundary, while the east and west is divided, in the Upper Austral, into the Carolinian and Upper Sonoran Zones. The Carolinian fauna is the most important, being dominant over approximately the eastern two-thirds of the state, except an irregular strip on the south which belongs to the Austroriparian. These faunal areas are becoming more clearly defined for the various species of birds, as the Biological Survey work is extended.

During the summer and fall of 1911, C. D. Bunker and T. A. Rocklund, of the University of Kansas Museum, in the interest of the State Biological Survey, made a wagon trip along the parts of the south, west, and north borders of the state, covering a distance of 1500 miles, while collecting in thirty-eight counties. The year was very unfavorable, in that it was exceedingly dry. In the western part of the state the vegetable life was entirely burned up, making the bird life unusually scarce. Nevertheless, about 1000 birds were collected, representing 84 species and varieties, 8 of which were new to the state. In 1912, from September to November, these same men worked the counties of Trego, Lane, and Gove, collecting some 700 birds.

Since the third edition of the A. O. U. check list has been published, the entire collection of birds in the University of Kansas has been gone over by Miss Arta Briggs, and nomenclature brought up to date and the subspecies of the Kansas birds carefully worked down. Twenty-seven species and subspecies were found that had not been recorded in Doctor Snow's Catalogue of Kansas Birds. The collection of birds in the University of Kansas Museum consists of 1519 mounted birds, and 7500 bird skins; comprising in all 722 species and subspecies. The greater part of these birds were collected in Kansas, yet there are birds of other states and countries in the collection. There are about 1100 skins from Yucatan, including one type (*Formicarius pallidus leucolæma*).

The study of Kansas birds, while not having been taken up by the government surveys, has been the subject of considerable work. Doctor Snow was the first scientist in the field. He began his study of Kansas birds upon his arrival at the University in 1866, and continued his interest until his death, September 20, 1908. He published his first edition of the Birds of Kansas in April of 1872, in the *Kansas Educational Journal*. He reported 259 species and varieties. In July of 1872 he published an amended list, containing 277 species and varieties, in the *Western Guide* of the Kansas Pacific Railroad. In October of the same year he issued a third edition, containing 282 species and varieties. In 1875 he contributed a fourth edition to the Kansas Academy of Science, with 302 birds listed. His fifth and last appeared in the May of 1903 edition of the Kansas Academy of Science, and contained 342 birds. In November of the same year he added three species that he had already listed, but had omitted from his last list. They were the Road Runner (*Geococcyx californianus*), Black-bellied Plover (*Squatarola squatarola*), and the Black-throated Green Warbler (*Dendroica virens*). In December, 1904, he added the Parasitic Jaejer (*Stercorarius parasiticus*), White-winged Crossbill (*Loxia leucoptera*), and the Grove-billed Ani (*Crotophaga sulcirostris*). In his fifth edition was listed the Logger-headed Shrike (*Lanius ludovicianus ludovicianus*), the southern form that does not inhabit Kansas; the Oregon Junco (*Junco hyemalis oreganus*), a bird of the West; and the Long-billed Marsh Wren (*Telmatodytes palustris palustris*) of the East. As no evidence could be found that these birds

had ever been taken in the state, and as Kansas is entirely out of their range, they are omitted from the following list, which is primarily based upon Doctor Snow's fifth edition of the Birds of Kansas. Revisions have been made where present-day nomenclature demanded, and the descriptive notes have been changed in some places, since the shifting conditions in bird life change with each decade, and more study has brought new information.

I. Family COLYMBIDÆ. Grebes.

1. Æchmophorus occidentalis (Lawr.). Western Grebe. Accidental; one collected in Shawnee county, November 8, 1901, by Robert Maxwell. In the University mounted collection.

2. Colymbus holbœlli (Reinhardt). Holbœll's Grebe. Occasional; added to the list October 29, 1910, by Logan Evans, Lawrence, Kan. In the University mounted collection, new to the list. Not in Doctor Snow's catalogue.

3. Colymbus auritus (Linn.). Horned Grebe. Migratory; rare. One collected in Nickerson county, November 15, 1906, by Mr. K. C. Beck. In museum skin collection.

4. Colymbus nigricollis californicus (Heerm.). Eared Grebe. Migratory.

5. Podilymbus podiceps (Linn.). Pied-billed Grebe. A rare summer resident; an abundant migrant.

II. Family GAVIIDÆ. Loons.

6. Gavia immer (Brunn.). Loon. Migratory; rare.

III. Family STERCORARIIDÆ. Skuas and Jaegers.

7. Stercorarius parasiticus (Linn.). Parasitic Jaejer. Accidental. "A young male of this species was captured along the Kansas river, near Lawrence, on October 10, 1908, by Banks Brown. This specimen was mounted by Leverett A. Adams, and is now in the Museum of the University of Kansas. This species not having been previously reported as 'seen' or 'likely to occur in Kansas,' is an absolute addition to our avifauna."—Quoted from Doctor Snow's Notes of 1904 on Birds of Kansas, published in volume XIX, Transactions of the Kansas Academy of Science, page 263.

IV. Family LARIDÆ. Gulls and Terns.

8. Larus argentatus (Portoppidan). Herring Gull. Migratory; rare.

9. Larus californicus (Lawr.). California Gull. Quoted as accidental by Doctor Snow. No Kansas collected specimen in the Kansas Museum.

10. Larus delawarensis (Ord). Ring-billed Gull. Migratory; frequent.

11. Larus atricilla (Linn.). Laughing Gull. Migratory; occasional. The first was collected in 1888 in Douglas county. Five more have been

collected during the years 1901, 1905, 1906 and 1911. These are now in the University Museum. Doctor Snow did not include this species in his catalogue.

12. Larus franklini (Rich.). Franklin's Gull. Migratory.

13. Larus philadelphia (Ord). Bonaparte's Gull. Migratory; rare.

14. Xema sabini (Sab.). Sabine's Gull. Quoted as accidental by Doctor Snow. No specimen of this species in the museum. A male was collected October 3, 1909, at Hamilton, Greenwood county, Kansas, by G. C. Rinker. Colonel Goss has collected this bird in Kansas.

15. Sterna forsteri (Nut.). Forster's Tern. Migratory; not uncommon. Recorded as common by Doctor Snow.

16. Sterna hirundo (Linn.). Common Tern. Migratory; rare.

17. Sterna antillarum (Lesson). Least Tern. Intermittently common as a summer resident.

18. Hydrochelidon nigra surinamensis (Gmel.). Black Tern. A rare summer resident; common in migration.

V. Family ANHINGIDÆ. Darters.

19. Anhinga anhinga (Linn.). Water Turkey. Catalogued as accidental by Doctor Snow. No Kansas collected specimen in the University Museum.

VI. Family PHALACROCORACIDÆ. Cormorants.

20. Phalacrocorax aurautus aurautus (Lesson). Double-crested Cormorant. Migratory; rather common.

21. Phalacrocorax vigua mexicanus (Brandt). Mexican Cormorant. Quoted as accidental by Doctor Snow. No specimen in the University Museum.

VII. Family PELECANIDÆ. Pelicans.

22. Pelicanus erythrorhynchos (Gmelin). White Pelican. Migratory; common.

VIII. Family FREGTIDÆ. Man-o'-war-birds.

23. Fregata aquila (Linn.). Man-o'-war-bird. Quoted as accidental by Doctor Snow. No Kansas collected specimen in the Museum.

IX. Family ANATIDÆ. Ducks, Geese, and Swans.

24. Mergus americanus (Cassin). Merganser. Winter sojourner; not uncommon.

25. Mergus serrator (Linn.). Red-breasted Merganser. This species was omitted in Doctor Snow's last catalogue, owing to the fact that during the last thirty years, *i. e.*, up to 1903, no specimen had been taken in the state. Two birds of this species were collected in the year of 1907; one by L. L. Dyche. It was found dead by him near Lake View, April 19, 1907. Dan Breese collected one in November of the same year in Sedgwick county. Both skins are in the University Museum.

26. Lophodytes cucullatus (Linn.). Hooded Merganser. A rare resident; a common migrant.

27. Anas platyrhynchos (Linn.). Mallard. A rare resident; a common migrant.

28. Anas rubripes (Brewster). Black Duck. Migratory; occasional. Recorded by Doctor Snow as "very rare." This duck has been collected in Douglas county—one specimen a year in 1899, 1901 and 1903, and in Pratt county in 1910. These specimens are in the University Museum. A specimen classified as Red-legged Black Duck by Mr. G. C. Rinker, of Hamilton, Greenwood county, Kansas, was collected by a friend of his, February 17, 1911, and loaned to the museum by him. *Anas rubripes tristis* is considered only in the hypothetical by the A. O. U. check list. The dried condition of the mounted specimen made absolute identification impossible by the museum systematists.

29. Anas fulvigula maculosa (Sennett). Mottled Duck. Migratory; rare.

30. Chaulelasmus streperus (Linn.). Gadwell. A rare summer resident; a common migrant.

31. Mareca americana (Gmel.). Baldpate. A rare summer resident; a common migrant.

32. Nettion carolinensis (Gmel.). Green-winged Teal. An abundant migrant; an occasional winter sojourner.

33. Querquedula discors (Linn.). Blue-winged Teal. An occasional summer resident; abundant in migration.

34. Querquedula cyanoptera (Vieill.). Cinnamon Teal. Migratory; rare in the eastern part of the state, not uncommon in the western part. Recorded as rare by Doctor Snow.

35. Spatula clypeata (Linn.). Shoveller. A rare summer resident; a common migrant.

36. Dafila acuta (Linn.). Pintail. Migratory; rare.

37. Aix sponsa (Linn.). Wood Duck. Summer resident; once common.

38. Marila americana (Eyton). Redhead. Migratory; common.

39. Marila valisineria (Wils.). Canvasback. Migratory; usually common.

40. Marila marila (Linn.). Scaup Duck. Migratory; rare.

41. Marila affinis (Eyton). Lesser Scaup Duck. Migratory; not common.

42. Marila collaris (Donov.). Ring-necked Duck. Migratory; once common.

43. Clangula clangula americana (Bonap.). Golden-eye. Migratory; rare.

44. Clangula islandica (Gmelin). Barrow's Golden-eye. This is a new species for the state. Doctor Snow did not include it in any of his lists, and according to the A. O. U. check list, Nebraska is the farthest south record for this species. Two were collected in Douglas county December 11, 1903, by Mr. Crow; one in Leavenworth county March 10, 1904, by Mr. Sheen; and two at Lake View, March 27, 1909, by L. L. Dyche, jr. These are mounted in the University Museum.

45. Charitonetta albeola (Linn.). Buffle-head. Migratory; common.

46. Harelda hyemalis (Linn.). Old squaw. Migratory; very rare. Quoting from Doctor Snow's catalogue: "This species is new to our fauna. Its capture was communicated to me by my friend, Dr. R. Matthews, of Wichita. The specimen is in his collection. It was captured by Dr. J. W.

Shultz, in November, 1901, on Patterson lake, in the south end of Harvey county. Mr. Jerald Volk, of Wichita, has another specimen, taken March 1, 1902, at Gantz Mill, in Pratt county. Both of these birds were alone when shot. Both "came in with a whiz and struck the water with such force as to attract attention."

47. Somateria v-nigra (Gray). Pacific Eider. Accidental. One specimen is in the University Museum.

48. Oidemia perspicillata (Linn.). Surf Scoter. Accidental. Added to the list in the year 1888 by A. L. Bennett. This specimen is in the University Museum. Arthur Marks killed four of these birds October 17th, 1913, near Lawrence.

49. Erismatura jamaicensis (Gmelin). Ruddy Duck. Rare; once common in migration.

50. Chen hyperboreus hyperboreus (Pallas). Lesser Snow Goose. Migratory; common.

51. Chen cærulescens (Linn.). Blue Goose. Migratory; rare.

52. Anser albifrons gambeli (Hart.). White-fronted Goose. Migratory; common.

53. Branta canadensis canadensis (Linn.). Canada Goose. Migratory; common.

54. Branta canadensis hutchinsi (Rich.). Hutchins's Goose. Migratory; common.

55. Branta bernicla glaucogastra (Brehm). Brant. Migratory; very rare.

56. Olor columbianus (Ord). Whistling Swan. Migratory; not uncommon.

57. Olor buccinator (Rich.). Trumpeter Swan. Migratory; rare.

X. Family PLATALEIDÆ. Spoonbills.

58. Ajaia ajaja (Linn.). Roseate Spoonbill. Accidental. Quoting from Doctor Snow's catalogue: "This species is new to the Kansas catalogue. It was communicated to me by Dr. R. Matthews, of Wichita, who wrote me September 26, 1899, that a male specimen was captured March 20, 1899, on Four-Mile creek, near Douglas, in Butler county. It became the property of Mr. Jerald Volk, of Wichita."

XI. Family IBIDIDÆ. Ibises.

59. Plegadis guarauna (Linn.). White-faced Glossy Ibis. Migratory; rare. One was collected in Douglas county October, 1879, by W. L. Bullene, and mounted by Frank Dixon. It is in the University Museum.

XII. Family CICONIIDÆ. Wood Ibises.

60. Mycteria americana (Linn.). Wood Ibis. Accidental. No Kansas collected specimen in the University collection.

XIII. Family ARDEIDÆ. Herons, Bitterns, Etc.

61. Botaurus lentiginosus (Montagu). Bittern. Summer resident; common.

62. Ixobrychus exilis (Gmelin). Least Bittern. Summer resident; not common.

63. Ardea herodias herodias (Linn.). Great Blue Heron. Summer resident; common.

64. Herodias egretta (Gmel.). Egret. Summer visitant; not uncommon.

65. Egretta candidissima candidissima (Gmelin). Snowy Egret. Summer visitant; occasional.

66. Florida cærulea (Linn.). Little Blue Heron. Summer visitant; rare.

67. Butorides virescens virescens (Linn.). Green Heron. Summer resident; common.

68. Nycticorax nycticorax nævius (Bood.). Black-crowned Night Heron. A rare summer resident.

69. Nyctanassa violacea (Linn.). Yellow-crowned Night Heron. A rare summer resident.

XIV. Family GRUIDÆ. Cranes.

70. Grus americana (Linn.). Whooping Crane. Migratory; rare.

71. Grus canadensis (Linn.). Little Brown Crane. Migratory; common.

72. Grus mexicana (Mull.). Sandhill Crane. Migratory; common.

XV. Family RALLIDÆ. Rails, Gallinules, and Coots.

73. Rallus elegans (Aud.). King Rail. Summer resident; common.

74. Rallus virginianus (Linn.). Virginia Rail. A very rare summer resident; a common migrant.

75. Porzana carolina (Linn.). Sora. Summer resident; common.

76. Coturnicops noveboracensis (Gmel.). Yellow Rail. Summer resident; rare.

77. Creciscus jamaicensis (Gmelin). Black Rail. Summer resident; rare.

78. Ionornis martinicus (Linn.). Purple Gallinule. Occasional. Two specimens have been collected in Douglas county; the first one by Grant Mull, April 26, 1896, and the second by H. J. Newman, April 18, 1909. Both are in the University Museum.

79. Gallinula galeata (Licht.). Florida Gallinule. A very rare summer resident.

80. Fulica americana (Gmelin). Coot. Summer resident; abundant.

XVI. Family PHALAROPODIDÆ. Phalaropes.

81. Phalaropus fulicarius (Linn.). Red Phalarope. Accidental. Added to the list by E. E. Brown, November 5, 1905. The specimen was a female, taken at Lake View.

82. Lobipes lobatus (Linn.). Northern Phalarope. Migratory; rare.

83. Steganopus tricolor (Vieillot). Wilson's Phalarope. Migratory; common.

XVII. Family RECURVIROSTRIDÆ. Avocets and Stilts.

84. Recurvirostra americana (Gmelin). American Avocet. A rare summer resident; a common migrant.

85. Himantopus mexicanus (Muller). Black-necked Stilt. Accidental. Added to the list in 1906 by T. H. Griffiths. Shot ten miles north of Wichita. Donated to the University; now in the F. H. Snow collection.

XVIII. Family SCOLOPACIDÆ. Snipes, Sandpipers, and Curlews.

86. Philohela minor (Gmelin). Woodcock. Probably extinct. Given by Doctor Snow as a rare summer resident.

87. Gallinago delicata (Ord). Wilson's Snipe. Migratory; common.

88. Macrorhamphus griseus scolopaceus (Say). Long-billed Dowitcher. Mirgatory; common.

89. Micropalma himantopus (Bonap.). Stilt Sandpiper. Migratory; common.

90. Tringa canutus (Linn.). Knot. Migratory; rare. An immature male was collected September 19, 1911, at Hamilton, Greenwood county, Kansas, by G. C. Rinker, and loaned to the Museum by him.

91. Pisobia maculata (Vieill.). Pectoral Sandpiper. Migratory; common.

92. Pisobia fuscicollis (Vieill.). White-rumped Sandpiper. Migratory; common.

93. Pisobia bairdi (Coues). Baird's Sandpiper. Migratory; rare.

94. Pisobia minutilla (Vieill.). Least Sandpiper. Migratory; abundant.

95. Pelidna alpina sakhalina (Vieill.). Red-backed Sandpiper. Migratory; rare. One female collected at Lawrence, Kan., by Isaac Morgan, May 11, 1885. In the University skin collection. An immature male collected October 13, 1912, by G. C. Rinker, another by him October 20, 1912, at Hamilton, Kan.

96. Ereunetes pusillus (Linn.). Semipalmated Sandpiper. Migratory; common.

97. Ereunetes mauri (Cabanis). Western Sandpiper. Migratory; occasional. New to the list, added by G. D. Hanna, May 26, 1906, near Lawrence. In the spring of 1906 six more specimens were collected, and one in 1910.

98. Calidris leucophæa (Pallas). Sanderling. Accidental.

99. Limosa fedoa (Linn.). Marbled Godwit. Migratory; occasional.

100. Limosa hæmastica (Linn.). Hudsonian Godwit. Migratory; rare.

101. Totanus melanoleucus (Gmelin). Greater Yellow-legs. Migratory; common.

102. Totanus flavipes (Gmelin). Yellow-legs. Migratory; abundant.

103. Helodromas solitarius solitarius (Wils.). Solitary Sandpiper. Migratory; common.

104. Catoptrophorus semipalmatus semipalmatus (Gmelin). Willet. Migratory; occasional. Not given in Doctor Snow's list. Two males were collected by Scott Griesa, May 6, 1909, in Douglas county, Kansas. One male was collected in Greenwood county by G. C. Rinker, September 5, 1912. It was in company with six Black-bellied Plovers.

105. Catoptrophorus semipalmatus inornatus (Brewster). Western Willet. Migratory; occasional. Given by Doctor Snow as "rare."

106. Bartramia longicauda (Bechst.). Bartramian Sandpiper. A common summer resident.

107. Tryrgites subruficollis (Vieill.). Buff-breasted Sandpiper. Migratory; rare.

108. Actitis macularia (Linn.). Spotted Sandpiper. A common summer resident; abundant migrant.

109. Mumenius americanus (Brechst.). Long-billed Curlew. A rare summer resident; a common migrant.

110. Numenius borealis (Forst.). Eskimo Curlew. Migratory; now nearly extinct.

XIX. Family CHARADRIIDÆ. Plovers.

111. Squatarola squatarola (Linn.). Black-bellied Plover. Migratory; occasional. This species was omitted from Doctor Snow's fifth edition of his Birds of Kansas Catalogue, as the actual capture of the bird had not been satisfactorily verified. In his Notes of 1903 on the Birds of Kansas, published in volume XIX, Kansas Academy of Science, he reported the capture of a bird of this species by Ed. Goldberg, near Wichita, in 1896. Doctor Matthews sent the mounted bird to Doctor Snow, May 22, 1903. L. A. Bennett, October 29, 1886, collected two specimens of this species near Lawrence. They are mounted and in the University Museum. One more bird of this species was collected in Douglas county, October 29, 1887. The latest records of this bird are found in G. C. Rinker's data of 1912. Two males and a female were collected by him September 8, 1912. Three others in the same flock were not saved. An immature female was taken by him October 13, 1912, at the same place, Hamilton, Kan.

112. Charadrius dominicus dominicus (Mull.). Golden Plover. Migratory; common.

113. Oxyechus vociferus (Linn.). Killdeer. An abundant summer resident.

114. Ægialitis semipalmata (Bonap.). Semipalmated Plover. Migratory; not common.

114½. Ægialitis meloda (Ord). Piping Plover. Rare. One collected May 27, 1909, by L. L. Dyche, Douglas county, Kansas. In University Museum.

115. Ægialitis nivosa (Cass). Snowy Plover. A rare summer resident in southwestern Kansas.

116. Podasocys montanus (Towns.). Mountain Plover. A common summer resident in the western part of the state.

XX. Family APHRIZIDÆ. Turnstone.

117. Arenaria interpres morinella (Linn.). Turnstone. Accidental Doctor Snow did not give the subspecies name, but it was doubtless *morinella*, as this is the only subspecies that ranges over the whole of North America. This species is in the University Museum. A male *morinella* was collected October 1, 1911, in Greenwood county, by G. C. Rinker. It was in company with two Wilson's Phalaropes. This one was loaned to the University for comparison.

XXI. Family ODONTOPHORIDÆ. Bob-whites.

118. Colinus virginianus virginianus (Linn.). Bob-white. An abundant resident.

119. Colinus virginianus texanus (Lawr.). Texas Bob-white. Listed by Doctor Snow as "a very rare resident in southwestern Kansas; formerly not uncommon." Now probably extinct. There are none in the University Museum collection.

XXII. Family TETRAONIDÆ. Grouse.

120. Bonasa umbellus umbellus (Linn.). Ruffed Crouse. Probably extinct. It has not been seen in the state for years.

121. Tympanuchus americanus americanus (Reid). Prairie Chicken. Formerly an abundant resident; still common in some parts of western Kansas.

122. Tympanuchus pallidicinctus (Ridg.). Lesser Prairie Chicken. A rare resident in southwestern Kansas.

123. Pediœcetes phasianellus campestris (Ridgw.). Prairie Sharp-tailed Grouse. Given by Doctor Snow as "a common resident in western Kansas." They were not seen during the fall and summer of 1911, when a survey of that part of the state was being made. There is no authentic Kansas collected specimen of this Grouse in the University Museum.

XXIII. Family MELEAGRIDÆ. Turkeys.

124. Meleagris gallopavo silvestris (Vieillot). Wild Turkey. Once an abundant resident in southern Kansas; now extremely rare if not already extinct.

XXIV. Family COLUMBIDÆ. Pigeons.

125. Ectopistes migratorius (Linn.). Passenger Pigeon. Now extinct from the state. Given by Doctor Snow as a rare summer resident; once abundant.

126. Zenaidura macroura carolinensis (Linn.). Mourning Dove. An abundant summer resident in the eastern part of the state.

127. Zenaidura macroura margnella (——). Common in western part of the state. A new bird to the list.

XXV. Family CATHARTIDÆ. American Vultures.

128. Cathartes aura septentrionalis (Wied). Turkey Vulture. An abundant summer resident.

129. Catharista urubu (Vieill.). Black Vulture. A very common summer resident.

XXVI. Family BUTEONIDÆ. Hawks, Eagles, Kites, Etc.

130. Elanoides forficatus (Linn.). Swallow-tailed Kite. An irregular summer resident.

131. Ictinia mississippiensis (Wils.). Mississippi Kite. A common summer resident in Barber and Comanche counties; irregularly nesting farther north.

132. Circus hudsonius (Linn.). Marsh Hawk. Resident; common.

133. Accipiter velox (Wils.). Sharp-shinned Hawk. A winter sojourner.

134. Accipiter cooperi (Bonap.). Cooper's Hawk. Resident; common.

135. Astur atricapillus atricapillus (Wils.). Goshawk. A very rare winter visitant.

136. Buteo borealis borealis (Gmelin). Red-tailed Hawk. Resident; common.

137. Buteo borealis krideri (Hoopes). Kreider's Hawk. A visitant.

138. Buteo borealis calurus (Cass). Western Red-tail. A not uncommon winter sojourner.

139. Buteo borealis harlani (Aud.). Harlan's Hawk. Occasional.

140. Buteo lineatus lineatus (Gmelin). Red-shouldered Hawk. Resident; common in the southern part of the state.

141. Buteo swainsoni (Bonap.). Swainson's Hawk. Resident; common.

142. Buteo platypterus (Vieillot). Broad-winged Hawk; rare.

143. Archibuteo lagopus sancti-johannis (Gmel.). Rough-legged Hawk. A common winter sojourner.

144. Archibuteo ferrugineus (Licht.). Ferruginous Rough-leg. Resident; common, except in eastern part.

145. Aquila chrysætus (Linn.). Golden Eagle. A rare resident; rather common in winter.

146. Haliæetus leucocephalus leucocephalus (Linn.). Bald Eagle. A rare resident.

XXVII. Family FALCONIDÆ. Falcons.

147. Falco rusticolus rusticolus (Linn.). Gray Gyrfalcon. Accidental. Added to the list in 1881 by Dr. C. P. Blachley.

148. Falco mexicanus (Schl.). Prairie Falcon. A rare resident.

149. Falco peregrinus anatum (Bonap.). Duck Hawk. Resident; rare.

150. Falco columbarius columbarius (Linn.). Pigeon Hawk. Migratory; rare.

151. Falco columbarius richardsoni (Ridgw.). Richardson's Pigeon Hawk. Migratory; rare in eastern, common in western Kansas.

152. Falco sparverius sparverius (Linn.). Sparrow Hawk. Resident; abundant.

153. Falco sparverius phalæna (Lesson). Desert Sparrow Hawk. Common in the western part of the state. New to the list. Not included in Snow's catalogue.

XXVIII. Family PANDIONIDÆ. Ospreys.

154. Pandion haliætus carolinensis (Gmelin). Osprey. A rare summer resident.

XXIX. Family ALUCONIDÆ. Barn Owls.

155. Aluco pratincola (Bonap.). Barn Owl. Resident.

XXX. Family STRIGIDÆ. Horned Owls, Etc.

156. Asio wilsonianus (Less.). Long-eared Owl. Resident; common.

157. Asio flammeus (Pontoppidan). Short-eared Owl. Resident; common.

158. Strix varia varia (Barton). Barred Owl. Resident; common.

159. Cryptoglaux acadica acadica (Gmelin). Saw-whet Owl. Winter sojourner; not common.

160. Otus asio asio (Linn.). Screech Owl. Resident; abundant.

161. Otus asio aikeni (Brewster). Aiken's Screech Owl. Common resident in western Kansas. A new species for the state; added to the list, in the fall of 1911, by C. D. Bunker and T. A. Rocklund.

162. Bubo virginianus virginianus (Gmelin). Great Horned Owl. Resident; common.

163. Bubo virginianus palescens (Stone). Western Horned Owl. Common in western part of state. Listed as accidental in Doctor Snow's catalogue.

164. Nyctea nyctea (Linn.). Snowy Owl. Winter sojourner; rare.

165. Speotyto cunicularia hypogæa (Bonap.). Burrowing Owl. Resident; abundant in central and western Kansas.

XXXI. Family PSITTACIDÆ. Paroquets.

166. Conuropsis carolinensis (Linn.). Carolina Paroquet. Formerly a common resident. Now extinct.

XXXII. Family CUCULIDÆ. Cuckoos, Anis, Etc.

167. Crotophaga sulcirostris (Swains). Grove-billed Ani. Accidental. "A specimen of this bird was captured by a farmer near Emporia about the first of November, 1904. It is in the collection of the State Normal School, and was reported to me [Doctor Snow] by Prof. L. C. Wooster of that institution."

168. Geococcyx californianus (Lesson). Road-runner. Occasional in the southwestern part of Kansas.

169. Coccyzus americanus americanus (Linn.). Yellow-belled Cuckoo. Summer resident; common.

170. Coccyzus erythrophthalmus (Wils.). Black-billed Cuckoo. A rare summer resident.

XXXIII. Family ALCEDINIDÆ. Kingfisher.

171. Ceryle alcyon (Linn.). Belted Kingfisher. Summer resident; common.

XXXIV. Family PICIDÆ. Woodpeckers.

172. Dryobates villosus villosus (Linn.). Hairy Woodpecker. Resident; common.

173. Dryobates pubescens medianus (Linn.). Downy Woodpecker. Resident; common.

174. Sphyrapicus varius varius (Linn.). Yellow-bellied Sapsucker. Migratory; occasional.

175. Sphyrapicus varius nuchalis (Baird). Red-napped Sapsucker. Migratory; rare.

176. Phlœotomus pileatus abieticola (Bangs). Pileated Woodpecker. Perhaps extinct. Given by Doctor Snow "as a very rare resident."

177. Melanerpes erythrocephalus (Linn.). Red-headed Woodpecker. A common summer resident.

178. Asyndesmus lewisi (Riley). Lewis's Woodpecker. A common visitant in western Kansas. One taken in Douglas county by C. D. Bunker.

179. Centurus carolinus (Linn.). Red-bellied Woodpecker. Resident; common.

180. Colaptes auratus luteus (Bangs). Northern Flicker. Resident; common.

181. Colaptes cafer collaris (Vigors.). Red-shafted Flicker. Resident; common in western Kansas, rare in the eastern part of the state.

XXXV. Family CAPRIMULGIDÆ. Nighthawks.

182. Antrostomus carolinensis (Linn). Chuck-will's-widow. Accidental. "A specimen of this bird was captured on the Arkansas river, south of Wichita, June 12, 1898, by Doctor Matthews, of Wichita, who submitted it to me for determination."—[Quoted from Doctor Snow's catalogue.] A female of the species was collected April 30, 1912, in the heavy wooded bottoms near Hamilton, by G. C. Rinker.

183. Antrostomus vociferus vociferus (Wils.). Whippoorwill. Summer resident; common.

184. Phalænoptilus nuttalli nuttalli (Aud.). Poorwill. A common summer resident.

185. Phalænoptilus nuttalli nitidus (Brewst.). Frosted Poorwill. A common summer resident.

186. Chordeiles virginianus virginianus (Gmel.). Nighthawk. A common summer resident in eastern Kansas.

187. Chordeiles virginianus henryi (Cass.). Western Nighthawk. A common summer resident in western Kansas.

188. Chordeiles virginianus sennetti (Coues). Sennett's Nighthawk. A not uncommon summer resident. Not included in Doctor Snow's lists.

XXXVI. Family MICROPOIDÆ. Swifts.

189. Chætura pelagica (Linn.). Chimney Swift. An abundant summer resident.

XXXVII. Family TROCHILIDÆ. Hummingbirds.

190. Archilochus colubris (Linn.). Ruby-throated Hummingbird. A common summer resident.

XXXVIII. Family TYRANNIDÆ. Tyrant Flycatchers.

191. Muscivora forficata (Gmelin). Scissor-tailed Flycatcher. A summer resident; not uncommon in southern Kansas. They were common in the south part of the state in 1911.

192. Tyrannus tyrannus (Linn.). Kingbird. Summer resident; abundant.

193. Tyrannus verticalis (Say). Arkansas Kingbird. A common summer resident.

194. Myiarchus crinitus (Linn.). Crested Flycatcher. An abundant summer resident in eastern Kansas.

195. Sayornis phœbe (Lath.). Phœbe. A common summer resident of eastern Kansas.

196. Sayornis sayus (Bonap.). Say's Phœbe. A common summer resident in western Kansas.

197. Nuttallornis borealis (Swain.). Olive-sided Flycatcher. A rare summer resident.

198. Myiochanes virens (Linn.). Wood Pewee. Summer resident; common.

199. Myiochanes richardsoni richardsoni (Swain.). Western Wood Pewee. A rare summer resident.

200. Empidonax virescens (Vieillot). Acadian Flycatcher. A not uncommon summer resident in eastern Kansas.

201. Empidonax trailli trailli (Aud.). Traill's Flycatcher. A not uncommon summer resident in eastern Kansas; common in western Kansas.

202. Empidonax trailli alnorum (Brewster). Alder Flycatcher. A rare summer resident.

203. Empidonax minimus (Baird). Least Flycatcher. Migratory; common.

XXXIX. Family ALAUDIDÆ. Larks.

204. Otocoris alpestris praticola (Henshaw). Prairie Horned Lark. A common summer resident in eastern Kansas.

205. Otocoris alpestris leucolæma (Coues). Desert Horned Lark. A common summer resident. Doctor Snow listed *arenicola*, but that form is now listed under *leucolæma*.

XL. Family CORVIDÆ. Magpies, Crows, Jays, Etc.

206. Pica pica hudsonia (Sab.). Magpie. Winter visitant in western Kansas; rare.

207. Cyanocitta cristata cristata (Linn.). Blue Jay. Resident; abundant.

208. Corvus corax sinuatus (Wagl.). Raven. Formerly a resident; now a rare visitant in western Kansas.

209. Corvus cryptoleucus (Couch). White-necked Raven. A rare visitant in western Kansas.

210. Corvus brachyrhynchos brachyrhynchos (Brehm.). Crow; abundant.

211. Nucifraga columbiana (Wilson). Clark's Nutcracker. Accidental.

212. Cyanocephalus cyanocephalus (Wied.). Pinon Jay. Accidental.

XLI. Family ICTERIDÆ. Blackbirds, Orioles, Etc.

213. Dolichonyx oryzivorus (Linn.). Bobolink. A rare summer resident; common in migration.

214. Molothrus ater ater (Bodd.). Cowbird. An abundant summer resident.

215. Xanthocephalus xanthocephalus (Bonap.). Yellow-headed Blackbird. A rare summer resident; abundant in migration.

216. Agelaius phœniceus phœniceus (Linn.). Red-winged Blackbird. Summer resident.

217. Agelaius phœniceus fortis (Ridgw.). Thick-billed Redwing. Common in migration.

218. Agelaius phœniceus arctolagus (——). Redwing. Migratory. New to list.

219. Agelaius phœniceus predatorius (——). Redwing. Migratory. New to list.

220. Sturnella magna magna (Linn.). Meadowlark. Abundant resident in eastern Kansas.

221. Sturnella neglecta (Aud.). Western Meadowlark. An abundant resident in central and western Kansas; not uncommon in eastern Kansas.

222. Icterus spurius (Linn.). Orchard Oriole. An abundant summer resident.

223. Icterus galbula (Linn.). Baltimore Oriole. A summer resident in the eastern part of the state.

224. Icterus bullocki (Swains.). Bullock's Oriole. Common in western part of Kansas.

225. Euphagus carolinus (Mull.). Rusty Blackbird. Winter sojourner. Common in migration.

226. Euphagus cyanocephalus (Wagl.). Brewster's Blackbird. An occasional resident in western Kansas. Common in migration.

227. Quiscalus quiscula æneus (Ridgw.). Bronze Grackle. Abundant.

XLII. Family FRINGILLIDÆ. Finches, Sparrows, Etc.

228. Hesperiphona vespertina vespertina (W. Cooper). Evening Grosbeak. A new bird for the list. One male collected by G. C. Rinker, December 12, 1910, at Hamilton, Kan. His notes state that a flock of twenty or twenty-five were seen, but being very shy, only one was shot.

229. Hesperiphona vespertina montana (Ridg.). Western Evening Grosbeak. Migratory; rare.

230. Carpodacus purpureus purpureus (Gmel.). Purple Finch. Winter sojourner; rare.

231. Carpodacus mexicanus frontalis (Say). House Finch. Rare. Added to the list in 1894 by V. L. Kellogg and H. W. Menke.

232. Loxia curvirostra minor (Brehm.). Crossbill. An occasional winter visitant. Noted by Doctor Snow.

233. Loxia curvirostra stricklandi (Ridgw.). Mexican Crossbill. Occasional winter visitant. Entered as *bendirei* in Doctor Snow's fifth catalogue, but as Mexican in previous lists.

234. Loxia leucopteres (Gmelin). White-winged Crossbill. An irregular winter visitant. Not given in Doctor Snow's lists, but added later.

235. Acanthis linaria linaria (Linn.). Red-poll. Winter visitant; rare.

236. Astragalinus tristis tristis (Linn.). Goldfinch. Resident; abundant.

237. Astragalinus tristis pallidus (Mearns). Pale Goldfinch. This is a new bird for the state. Several birds of this species were collected March 12, 1910, by Alex Wetmore, near Lawrence.

238. Spinus pinus (Wilson). Pine Siskin. Winter resident; common.

239. Plectrophenax nivalis nivalis (Linn.). Snow Bunting. Winter visitant; rare.

240. Calcarius lapponicus lapponicus (Linn.). Lapland Longspur. An irregular and sometimes abundant winter sojourner.

241. Calcarius lapponicus alascensis (Ridgw.). Alaskan Longspur. Occasional in winter. Reported in the A. O. U. check list as wintering in western Kansas. Seven birds of this species were collected in December of 1908, in Douglas county, by C. D. Bunker and T. A. Rocklund.

242. Calcarius pictus (Swains.). Smith's Longspur. Winter sojourner; common.

243. Calcarius ornatus (Towns.). Chestnut-collared Longspur. Resident.

244. Rhynchophanes mccowni (Lawr.). MaCown's Longspur. A common winter sojourner in western Kansas.

245. Poœcetes gramineus gramineus (Geml.). Vesper Sparrow. A rare summer resident; a common migrant.

246. Poœcetes gramineus confinis (Baird). Western Vesper Sparrow. Migratory in the western part of the state. New to the list.

247. Passerculus sandwichensis savanna (Wils.). Savanna Sparrow. Migratory and abundant.

248. Passerculus sandwichensis alaudinus (Bonap.). Western Savanna Sparrow. Migratory; not uncommon.

249. Ammodramus savannarum australis (Maynard). Grasshopper Sparrow. Abundant.

250. Ammodramus savannarum bimaculatus (Swains.). Western Grasshopper Sparrow. Summer resident; abundant. New to the list.

251. Passerherbulus henslowi henslowi (Aud.). Henslow's Sparrow. Summer resident; rare.

252. Passerherbulus lecontei (Aud.). Leconte's Sparrow. Migratory; common.

253. Passerherbulus nelsoni nelsoni (Allen). Nelson's Sparrow. Listed by Doctor Snow as a rare summer resident. No specimen of this species in the University Museum.

254. Chondestes grammacus grammacus (Say). Lark Sparrow. Summer resident; abundant.

255. Chondestes grammacus strigatus (Swains.). Western Lark Sparrow. Common summer resident in central and western part of the state. New to the list.

256. Zonotrichia querula (Nutt.). Harris's Sparrow. Winter sojourner; abundant.

257. Zonotrichia leucophrys leucophrys (Forst.). White-crowned Sparrow. Migratory; common.

258. Zonotrichia leucophrys gambel (Nutt.). Gambel's Sparrow. Migratory; common. Given in Doctor Snow's list as "Intermedia"; that form is not recognized by the A. O. U. check list now.

259. Zonotrichia albicollis (Gmelin). White-throated Sparrow. Migratory; common.

260. Spizella monticola monticola (Gmel.). Tree Sparrow. Winter resident; abundant.

261. Spizella monticola ochracea (Brewst.). Western Tree Sparrow. Not an uncommon winter sojourner in western Kansas.

262. Spizella passerina passerina (Brecht.). Chipping Sparrow. Summer resident; common in eastern Kansas.

263. Spizella pallida (Swains.). Clay-colored Sparrow. Migratory; not uncommon.

264. Spizella pusilla pusilla (Wils.). Field Sparrow. Summer resident; common in eastern Kansas.

265. Spizella pusilla arenacea (Chadbourne). Western Field Sparrow. The one Kansas collected specimen of the species in the University Museum was taken near Lawrence, October 23, 1874, by W. Osborn. This specimen is in Doctor Snow's collection, but is not listed in his catalogue.

266. Junco aikeni (Ridgw.). White-winged Junco. A rare winter visitant in western Kansas. A male and female were collected January 1, 1911, in Lawrence, Kan., by C. D. Bunker and Logan Evans.

267. Junco hyemalis hyemalis (Linn.). Slate-colored Junco. Winter sojourner; abundant.

268. Junco hyemalis connectens (Coues). Shufeld's Junco. Accidental. One specimen collected at Lawrence December 10, 1902, by John Rowley, and is now in the University collection.

269. Junco hyemalis montanus (Ridgw.). Montana Junco. One immature female collected by C. D. Bunker and F. A. Wetmore near Lawrence, December 1, 1907, is in the University Museum.

270. Peucæa cassini (Woodhouse). Cassin's Sparrow. Listed by Doctor Snow as a common summer resident in western Kansas. The one specimen of this bird was collected in Wallace county, June 18, 1883, and is now in the University Museum.

271. Melospiza melodia melodia (Wilson). Song Sparrow. A not uncommon winter sojourner and common in migration.

272. Melospiza melodia juddi (Bishop). Dakota Song Sparrow. Migratory in the western part of the state. Several collected in the fall of 1911 and 1912 by Bunker and Rocklund in the western part of the state.

273. Melosipiza lincolni lincolni (Aud.). Lincoln's Sparrow. Migratory; common.

274. Melosipiza georgiana (Lath.). Swamp Sparrow. A rare winter sojourner in eastern Kansas; common in migration.

275. Passerella iliaca iliaca (Merr.). Fox Sparrow. Winter sojourner; abundant in eastern Kansas, rare in western.

276. Passerello iliaca schistacea (Baird). Slate-colored Fox Sparrow. Listed by Doctor Snow as a rare winter visitant. No Kansas collected specimen of this bird in the University Museum.

277. Pipilo erythrophthalmus erythrophthalmus (Linn.). Towhee. Resident, except in severe winters; common in eastern Kansas.

278. Pipilo maculatus arcticus (Swains.). Arctic Towhee. Winter sojourner; rare in eastern and western Kansas.

279. Cardinalis cardinalis cardinalis (Linn.). Cardinal. Resident; common.

280. Zamelodia ludoviciana (Linn.). Rose-breasted Grosbeak. Summer resident; common.

281. Zamelodia melanocephala (Swains.). Black-headed Grosbeak. Summer resident; common in western half of the state.

282. Guiraca cærulea lazula (Lesson). Western Blue Grosbeak. A common summer resident in central and western Kansas.

283. Passerina cyanea (Linn.). Indigo Bunting. A common summer resident in eastern Kansas.

284. Passerina amœna (Say). Lazuli Bunting. A rare summer resident in western Kansas.

285. Passerina ciris (Linn.). Painted Bunting. A common summer resident in southeastern Kansas.

286. Spiza americana (Gmelin). Dickissel. Summer resident; abundant in eastern and central Kansas.

287. Calamospiza melanocorys (Stejn.). Lark Bunting. A common summer resident in western Kansas.

288. Passer domesticus (Linn.). English Sparrow. Resident; abundant.

XLIII. Family TANGARIDÆ. Tanagers

289. Piranga ludoviciana (Wils.). Western Tanager. Casual in western Kansas. Given by Doctor Snow as accidental.

290. Piranga erythromelas (Vieill.). Scarlet Tanager. An occasional summer resident.

291. Piranga rubra rubra (Linn.). Summer Tanager. Summer resident; common eastern Kansas.

XLIV. Family HIRUNDINIDÆ. Swallows.

292. Progne subis subis (Linn.). Purple Martin. Summer resident; common.

293. Petrochelidon lunifrons lunifrons (Say). Cliff Swallow. Summer resident; common.

294. Hirundo erythrogastra (Badd.). Barn Swallow. Summer resident; common.

295. Iridoprocne bicolor (Vieill.). Tree Swallow. A rare summer resident; common in migration.

296. Riparia riparia (Linn.). Bank Swallow. Summer resident; common.

297. Stelgidopteryx serripennis (Aud.). Roughed-winged Swallow. A common summer resident.

XLV. Family BOMBYCILLIDÆ. Waxwings.

298. Bombycilla garrula (Linn.). Bohemian Waxwing. Winter visitant; rare.

299. Bombycilla cedrorum (Vieill.). Cedar Waxwing. Winter resident; varying from rare to abundant in different years.

XLVI. Family LANIIDÆ. Shrikes.

300. Lanius borealis (Vieill.). Northern Shrike. A common winter sojourner.

301. Lanius ludovicianus excubitorides (Swains.). White-rumped Shrike. Summer resident.

302. Lanius ludovicianus migrans (W. Palmer). Migrant Shrike. Occasional. New to the list.

XLVII. Family VIREONIDÆ. Vireos.

303. Vireosylva olivacea (Linn.). Red-eyed Vireo. Summer resident; common.

304. Vireosylva gilva gilva (Vieill.). Warbling Vireo. Summer resident; common.

305. Lanivireo falavifrons (Vieill.). Yellow-throated Vireo. Summer resident; less common.

306. Lanivireo solitarius solitarius (Wilson). Blue-headed Vireo.

307. Vireo atricapillus (Woodhouse). Black-capped Vireo. A doubtful resident in southern Kansas. None seen by the museum collectors on their 1911 trip through that part of the state. One specimen in the University collection taken in Comanche county, 1885.

308. Vireo griseus griseus (Bodd.). White-eyed Vireo. A common summer resident.

309. Vireo belli belli (Aud.). Bell's Vireo. Summer resident; common.

XLVIII. Family MNIOTILTIDÆ. Wood Warblers.

310. Mniotilta varia (Linn.). Black-and-white Warbler. Summer resident; not uncommon.

311. Protonotaria citrea (Bodd.). Protonotary Warbler. A common summer resident in eastern Kansas.

312. Helmitheros vermivorus (Gmelin). Worm-eating Warbler. Migratory; rare.

313. Vermivora pinus (Linn.). Blue-winged Warbler. Visitant in migration.

314. Vermivora rubricapilla rubricapilla (Wilson). Nashville Warbler. Rare and migratory.

315. Vermivora celata celata (Say). Orange-crowned Warbler. Migratory; common.

316. Vermivora celata orestera. Western Orange-crowned Warbler. Migratory in the western part of the state.

317. Vermivora peregrina (Wils.). Tennessee Warbler. Migratory; rare.

318. Compsothlypis americana usneæ (Brewst.). Northern Parula Warbler. Common in migration.

319. Dendroica æstiva æstiva (Gmel.). Yellow Warbler. Summer resident; abundant.

320. Dendroica cærulescens cærulescens (Gmel.). Black-throated Blue Warbler. Migratory; rare.

321. Dendroica coronata (Linn.). Myrtle Warbler. A sojourner in open winters; common in migration.

322. Dendroica auduboni auduboni (Towns.). Audubon's Warbler. Migratory; common in western Kansas.

323. Dendroica magnolia (Wils.). Magnolia Warbler. Migratory; rare.

324. Dendroica cerulea (Wils.). Cerulean Warbler. A rare summer resident; common in migration.

325. Dendroica pensylvanica (Linn.). Chestnut-sided Warbler. Migratory; rare.

326. Dendroica striata (Forst.). Black-poll Warbler. Migratory; common.

327. Dendroica fusca (Muller). Blackburnian Warbler. Migratory; rare.

328. Dendroica virens (Gmel.). Black-throated Green Warbler. "I received a fragmentary skin of this species which had been identified by Mr. J. A. Allen, of the American Museum of Natural History, from Mr. F. F. Crevecœur, of Onaga, Pottawatomie county, Kansas, who states, 'It was shot, as I can remember, in 1890, on French creek, three miles north of Onaga.' This species was included in the early editions of my catalogue of birds of Kansas upon the authority of Prof. S. J. Baird."—Quoted from Doctor Snow's Notes for 1903 on Birds of Kansas, vol. XIX, Transactions of the Kansas Academy of Science.

329. Dendroica vigorsi (Aud.). Pine Warbler. A rare migrant in eastern Kansas.

330. Dendroica palmarum palmarum (Gmelin). Palm Warbler. A rare migrant in eastern Kansas.

331. Dendroica discolor (Vieill.). Prairie Warbler. A rare summer resident in eastern Kansas.

332. Seiurus aurocapillus (Linn.). Oven-bird. A common summer resident in eastern Kansas.

333. Seiurus noveboracensis notabilis (Ridgw.). Grinnell's Water Thrush. Migratory and rare.

334. Sieurus motacilla (Vieill.). Louisiana Water Thrush. Summer resident.

335. Oporornis formosus (Wils.). Kentucky Warbler. Summer resident in eastern Kansas; common.

336. Oporornis philadelphia (Wils.). Mourning Warbler. Migratory; rare.

337. Geothlypis trichas brachidactyla (Linn.). Northern Yellow-throat. A common resident.

338. Icteria virens virens (Linn.). Yellow-breasted Chat. Common summer resident.

339. Icteria virens longicauda (Lawr.). Long-tailed Chat. A not uncommon summer resident in western Kansas.

340. Wilsonia citrina (Bood.) Hooded Warbler. A rare summer resident in eastern Kansas.

341. Wilsonia pusilla pusilla (Wilson). Wilson's Warbler. Migratory; common.

342. Wilsonia canadensis (Linn.). Canada Warbler. Migratory in eastern Kansas; rare.

343. Setophaga ruticilla (Linn.). Redstart. Summer resident in eastern Kansas; common.

XLIX. Family MOTACILLIDÆ. Wagtails.

344. Anthus rubescens (Tunstall). Pipit. Migratory; common.

345. Anthus spraguei (Aud.). Sprague's Pipit. Migratory; rare in eastern and common in western Kansas.

L. Family MIMIDÆ. Thrashers, Mockingbirds, Etc.

346. Mimus polyglottos polyglottos (Linn.). Mockingbird. Common summer resident in the eastern part of the state.

347. Mimus polyglottos leucopterus (Vigors). Western Mockingbird. Common in western part of the state; summer resident. New to the list.

348. Dumetella carolinensis (Linn.). Catbird. Summer resident; common.

349. Toxostoma rufum (Linn.). Brown Thrasher. An abundant summer resident.

LI. Family TROGLODYTIDÆ. Wrens.

350. Salpinctes obsoletus obsoletus (Say). Rock Wren. Summer resident in central and western Kansas.

351. Thryothorus ludovicianus ludovicianus (Latham). Carolina Wren. Resident; common.

352. Thryomanes bewicki bairdi (Salv. & God.) Baird's Wren. A not uncommon resident in southwestern Kansas.

353. Thryomanes bewicki crytus (Oberholdster). Texas Wren. Common in southwestern part of Kansas. Not included in Doctor Snow's lists.

354. Troglodytes ædon parkmani (Aud.). Western House Wren. Summer resident; common. Given by Doctor Snow as *aztectus*.

355. Nannus hiemalis hiemalis (Vieill.). Winter Wren. Winter sojourner; not uncommon.

356. Cistothorus stellarus (Neuman). Short-billed Marsh Wren. Migratory; rare.

357. Telmatodytes palustris iliacus (Ridgw.). Prairie Marsh Wren. A rare summer resident. Listed in previous catalogues as *Telmatodytes polustris* (Wils.): Long-billed Marsh Wren; but this bird does not belong to this range.

LII. Family CERTHIIDÆ. Creepers.

358. Certhia familiaris americana (Bonap.). Brown Creepers. Common winter sojourner.

LIII. Family SITTIDÆ. Nuthatches.

359. Sitta carolinensis carolinensis (Latham). White-breasted Nuthatch. Resident; common.

360. Sitta canadensis (Linn.). Red-breasted Nuthatch. Migratory; rare.

LIV. Family PARIDÆ. Titmouse.

361. Bælophus bicolor (Linn.). Tufted Titmouse. Resident; abundant.

362. Penthestes atricapillus atricapillus (Linn.). Chickadee. Resident; common in the eastern part of the state.

363. Penthestes atricapillus septentrionalis (Harris). Long-tailed Chickadee. A common resident.

LV. Family SYLVIIDÆ. Kinglets, Gnatcatchers.

364. Regulus satrapa satrapa (Licht.). Golden-crowned Kinglet. A common winter sojourner.

365. Regulus calendula calendula (Linn.). Ruby-crowned Kinglet. A common winter sojourner.

366. Pilioptila cærulea cærulea (Linn.). Blue-gray Gnatcatcher. A rare summer resident; common in migration.

LVI. Family TURDIDÆ. Thrushes, Robins, and Bluebirds.

367. Myadestes townsendi (Aud.). Townsend's Solitaire. An occasional fall and winter visitant.

368. Hylocichla mustelina (Gmelin). Wood Thrush. A common summer resident.

369. Hylocichla fuscescens salicicola (Ridg.). Willow Thrush. Migratory; rare. New to the list.

370. Hylocichla aliciæ aliciæ (Baird). Grey-cheeked Thrush. Migratory; rare.

371. Hylocichla ustulata swainsoni (Tschudi). Olive-backed Thrush. Migratory; common.

372. Hylocichla guttata pallasi (Cabinis). Hermit Thrush. Migratory; rare.

373. Hylocichla guttata auduboni (Baird). Rocky Mountain Hermit Thrush.

374. Planesticus migratorius migratorius (Linn.). Robin. Resident; abundant.

375. Planesticus migratorius propinquus (Ridg.). Western Robin. A rare winter visitant in western Kansas. Accidental in eastern Kansas.

376. Planesticus migratorius achrusterus (Batch.). Southern Robin. Seven were collected in Douglas county in the spring of 1901. New to the list and to the state.

377. Ixoreus nævius nævius (Gmelin). Varied Thrush. Accidental. Added to the list in 1894 by V. L. Kellogg and H. W. Menke.

378. Sialia sialis sialis (Linn.). Bluebird. A common resident in eastern and central Kansas; rare in western Kansas.

379. Sialia currucoides (Bechstein). Mountain Bluebird. Occasional in western Kansas. Given by Doctor Snow as "a common winter sojourner in western Kansas; rare in eastern Kansas."

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 6, January, 1913.

(Whole Series, Vol. XVII, No. 6.)

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ANNOTATED LIST OF THE DIPLOPODA AND CHILOPODA, WITH A KEY TO THE MYRIAPODA OF KANSAS. . . . . *Horace Gunthorp.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 6] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No. 6

ANNOTATED LIST OF THE DIPLOPODA AND CHILOPODA, WITH A KEY TO THE MYRIAPODA OF KANSAS.

BY HORACE GUNTHORP.

Plate XX.

(Contribution from the Zoölogical Laboratory, No. 205.)

Submitted in partial fulfillment of the requirements for the degree of Master of Arts.

FOR some time past local forms of millipedes and centipedes have been accumulating in the museum of the University of Kansas. During the past three years their number has been greatly augmented by the collections made on the summer trips of the State Biological Survey, and this in spite of the fact that the extreme dry weather during all of these summers made the finding of such forms very difficult. This condition probably accounts for the almost total absence of members of the family Craspedosomidæ. During the summer of 1910 the survey party started in Trego county, in western Kansas, and then traveled east to Lawrence. The following year the route ran south from Lawrence to the southeast corner of the state, then west seventy-five miles, and then north again to the starting point. The route taken in 1912 was southwest from Lawrence to Elk and Butler counties, returning north to Junction City, in Geary county. Besides the work of the survey, a large part of the collecting has been done by the writer in Cowley, Douglas and Jefferson counties. Also, the collection has been added to by Mr. R. D. Lindsey from Labette and Montgomery counties, by Mr. E. H. Taylor from Anderson county, and by Mr. G. D. Hanna from various points in the state.

Received for publication September 25, 1912.

As practically no records of these classes have been made from Kansas since the work of Cragin in 1885, which was of a very fragmentary character, Dr. C. E. McClung suggested to me that I work over the material on hand. The following paper is the result, and gives the list of species collected in Kansas now in the museum. Also, mention is made of all other species known to me to have been previously recorded from the state. Undoubtedly this list does not include all the species that will be found to occur in the state, and it is my intention to publish a future paper giving the results of further collecting.

I have not attempted to give a list of synonyms under each species, but simply a reference to the original description and to the latest change in name, where necessary. The grouping into orders and families follows the writings of Dr. O. F. Cook for the class Diplopoda, and of Dr. Ralph V. Chamberlain for the class Chilopoda. I wish to thank Professor McClung for the kind assistance he has given me in the classifying of this material, and in the preparation of this paper.

Class DIPLOPODA.

Order MEROCHETA.

Family XYSTODESMIDÆ.

1. **Fontaria virginiensis** (Drury).

Julus virginiensis Drury. Ins. Exot., I, t. XLIII, fig. 8 (1770).
Fontaria virginiensis Gray. In Griffith's Animal Kingd., Ins. I, t. CXXXV, fig. 1 (1832).

A few of the specimens are white in color, with a dark slate-colored median line extending from the third or fourth segment to the penultimate. I judge these to be freshly moulted specimens, as there seem to be no specific differences otherwise.

Habitat: Anderson Co. (E. H. Taylor); Cowley Co. (writer); Greenwood, Labette and Montgomery counties (Biol. Survey); Wyandotte Co. (G. D. Hanna). Also reported from Shawnee Co. by Cragin.

Family CHELODESMIDÆ.

2. **Leptodesmus hispidipes** (Wood).

Polydesmus hispidipes Wood. Proc. Phila. Acad., 1864, p. 7.
Leptodesmus hispidipes Bollman. Entom. Americ., vol. IV, p. 2 (1888).

Habitat: Cowley, Douglas and Jefferson counties (writer).

3. **Leptodesmus placidus** (Wood).

Polydesmus placidus Wood. Proc. Phila. Acad., 1864, p. 9.
Leptodesmus placidus Bollman. Bull. 46, U. S. Natl. Mus., p. 122 (1893).

Recorded from Jefferson county under the name *Polydesmus floridus*, by Cragin.

4. **Oxidus gracilis** (C. L. Koch).

Fontaria gracilis Koch. Syst. der Myriop., p. 142 (1847).
Oxidus gracilis Cook. Proc. U. S. Natl. Mus., vol. XL, p. 631 (1911).

Collected in greenhouses. It is probably distributed throughout the state.

HABITAT: Winfield, Cowley Co. (A. D. Shaftesbury); Lawrence, Douglas Co. (G. D. Hanna).

FAMILY POLYDESMIDÆ.

5. **Polydesmus pinetorum** (Bollman).

Polydesmus pinetorum Bollman. Entom. Americ., vol. IV, p. 3 (1888).

HABITAT: Anderson Co. (E. H. Taylor); Bourbon and Butler Cos. (Biol. Survey); Cowley Co. (writer); Douglas Co. (E. H. Taylor and writer); Franklin Co. (Biol. Survey); Jefferson Co. (writer); Pottawatomie and Shawnee Cos. (Biol. Survey); Sumner Co. (writer).

6. **Polydesmus serratus** (Say).

Polydesmus serratus Say. Jour. Phila. Acad., vol. II, p. 106 (1821).

HABITAT: Coffey Co. (Biol. Survey).

7. **Scytonotus granulatus** (Say).

Polydesmus granulatus Say. Jour. Phila. Acad., vol. II, p. 107 (1821).
Scytonotus granulatus Bollman. Bull. 46, U. S. Natl. Mus., p. 146 (1893).

This seems to be one of the rarer millipeds, as there are only three specimens in the collection.

HABITAT: Douglas Co. (writer); Shawnee Co. (G. D. Hanna).

ORDER COELOCHETA.

FAMILY CRASPEDOSOMIDÆ.

8. **Cleidogona** sp.?

Cleidogona Cook & Collins. Ann. N. Y. Acad. Sci., vol. IX, p. 41 (1896).

I find in the collection a single specimen of this genus. As it is a female, I am unable to refer it to a species, as the specific characters are very largely based on the form of the male copulatory appendages.

HABITAT: Douglas Co. (G. D. Hanna).

FAMILY LYSIOPETALIDÆ.

9. **Lysiopetalum lactarium** (Say).

Julus lactarium Say. Jour. Phila. Acad., vol. II, p. 104 (1821).
Lysiopetalum lactarium Packard. Amer. Natur., vol. XVII, p. 555 (1883).

One specimen collected was found feeding on the crushed abdomen of a freshly killed cricket.

HABITAT: Anderson Co. (E. H. Taylor); Bourbon, Chase and Cherokee Cos. (Biol. Survey); Cowley and Douglas Cos. (writer); Elk, Graham and Greenwood Cos. (Biol. Survey); Jefferson Co. (writer); Marion, Osage, Pottawatomie and Riley Cos. (Biol. Survey); Sumner Co. (W. O. Riley); Trego Co. (Biol. Survey).

ORDER ANOCHETA.

FAMILY SPIROBOLIDÆ.

10. **Arctobolus marginatus** (Say).

Julus marginatus Say. Jour. Phila. Acad., vol. II, p. 105 (1821).
Arctobolus marginatus Cook. Harr. Alaska Exped., vol. VIII, p. 65 (1904).

HABITAT: Anderson Co. (E. H. Taylor); Bourbon Co. (Biol. Survey); Butler Co. (E. C. Harrah); Cherokee and Coffey Cos. (Biol. Survey); Cowley Co. (writer); Douglas Co. (G. D. Hanna and writer); Jefferson Co. (writer); Labette and Linn Cos. (Biol. Survey); Montgomery Co. (R. D. Lindsey); Wyandotte Co. (G. D. Hanna).

11. **Tylobolus uncigerus** (Wood).

Spirobolus uncigerus Wood. Proc. Phila. Acad., 1864, p. 15.
Tylobolus uncigerus Cook. Harr. Alaska Exped., vol. VIII, p. 67 (1904).

Under the name *Spirobolus uncigerus*, this species is recorded from Shawnee county by Cragin. Other recorded localities for this species are restricted to California and Oregon, so I do not believe it occurs in Kansas.

ORDER ZYGOCHETA.

FAMILY PARAJULIDÆ.

12. **Parajulus diversifrons** (Wood).

Julus diversifrons Wood. Proc. Phila. Acad., 1867, p. 43.
Parajulus diversifrons Bollman. Bull. 46, U. S. Natl. Mus., p. 119 (1893).

Some of the specimens referred to this species are females, and doubtfully *P. diversifrons*, but each locality is represented by well-defined males.

HABITAT: Cowley Co. (writer); Douglas Co. (G. D. Hanna); Norton Co. (C. D. Bunker).

13. **Parajulus impressus** (Say).

Julus impressus Say. Jour. Phila. Acad., vol. II, p. 102 (1821).
Parajulus impressus Bollman. Bull. 46, U. S. Natl. Mus., p. 183 (1893).

HABITAT: Clay Co. (W. E. Wolking); Coffey Co. (Biol. Survey); Cowley Co. (writer); Douglas Co. (G. D. Hanna and writer); Franklin and Graham Cos. (Biol. Survey); Jefferson Co. (writer); Labette and Montgomery Cos. (Biol. Survey); Sumner Co. (writer); Trego Co. (Biol. Survey).

14. **Parajulus venustus** (Wood).

Julus venustus Wood. Proc. Phila. Acad., 1864, p. 10.
Parajulus venustus Bollman. Bull. 46, U. S. Natl. Mus., p. 183 (1893).

There is a single male specimen in the collection. It agrees well with Wood's description and figures.

HABITAT: Winfield, Cowley Co. (writer, Mar. 1911). Also recorded from "Kansas" by Bollman.

Class CHILOPODA.

ORDER SCHIZOTARSIA.

15. **Scutigera forceps** (Rafinesque).

Selista forceps Rafinesque. Ann. of Nature, vol. I, p. 7 (1820).
Scutigera forceps Meinert. Proc. Amer. Phil. Soc., vol. XXI, p. 171 (1885).

This species is undoubtedly distributed throughout the greater part of the state.

HABITAT: Cowley Co. (writer); Douglas Co. (D. H. Wenrick and writer); Montgomery Co. (Biol. Survey); Sumner Co. (writer). It is also reported by Cragin from Barber, Labette and Shawnee counties.

ORDER ANAMORPHA.

FAMILY LITHOBIIDÆ.

16. **Lithobius forficatus** (Linnæus).

Scolopendra forficata Linnæus. Syst. Nat., I, p. 638 (1758).
Lithobius forficatus Leach. Trans. Linn. Soc., vol. XI (1815).

In the adult specimens the coxal pores vary from 5, 5, 5, 4 to 6, 6, 7, 5. This number is smaller than usually assigned to this species.

HABITAT: Cowley Co. (writer); Labette Co. (R. D. Lindsey); Sumner Co. (W. O. Riley).

17. **Lithobius jowensis** Meinert.

Lithobius jowensis Meinert. Proc. Amer. Phil. Soc., vol. XXI, p. 177 (1885).

HABITAT: Anderson Co. (E. H. Taylor); Bourbon and Coffey Cos. (Biol. Survey); Cowley and Douglas Cos. (writer); Franklin and Graham Cos. (Biol. Survey); Jefferson Co. (writer); Labette, Linn and Pottawatomie Cos. (Biol. Survey); Reno Co. (G. D. Hanna); Sumner Co. (W. O. Riley); Trego Co. (Biol. Survey).

18. **Lithobius kansensis,** n. sp.

General color light brown; posterior segments lighter. Legs and sterna yellow.

Angles of the eleventh and thirteenth dorsal plates produced, but not very sharply. Those of the eleventh are sometimes almost straight.

Head rather small, wider than long (2.7:2.4), almost round in shape, the posterior margin being less convex than the anterior margin or sides, thus making the posterior part of the head wider than the anterior. A median groove in front, from which a plainly marked furrow extends posteriorly for about a fourth of the length of the head. Slightly pilose.

Antennæ short, 2 mm. in length; composed of from 28 to 32 articles. All the articles short except the first two and the last one; rather densely pilose.

Ocelli nine to thirteen, arranged in three series; prominent.

Prosternal teeth 2 + 2, large.

Pores on the ventral surface of the coxæ of the last four pairs of legs numbering 2, 3, 3, 3, to 3, 4, 4, 4; all round, the proximal being the smallest.

Spines on the ventral surface of the first legs, 0, 0, 1 in number; of the penultimate legs, 1, 3, 3, 1, and armed at the end with two claws; of the anal legs, 1, 3, 1, 0, with two claws at the end.

Claw of the female genitalia wide, distinctly tripartite, the middle lobe the longest; basal spines (two on each side) short, stout, the inner the shorter.

Length, 8-10 mm.

HABITAT: Cowley and Douglas Cos. (writer). The type specimens were collected by the writer at Lawrence, Douglas county, during April, 1912. Most of the specimens were found under the bark of old logs in damp weather.

This species is closely related to *L. cantabrigensis* Meinert, but differs in respect to the number of articles in the antennæ, the number of coxal pores, and the number of eyes.

19. **Lithobius transmarinus** (L. Koch).

Lithobius transmarinus Koch. Die. Myriap. Lith., p. 33 (1862).

This is by far the most common species of *Lithobius* in the state. It is found under almost anything that will give shelter. It is an especially good dry-weather form.

HABITAT: Anderson Co. (E. H. Taylor); Bourbon, Chase and Cherokee Cos. (Biol. Survey); Cowley and Douglas Cos. (writer); Graham and Greenwood Cos. (Biol. Survey); Jefferson Co. (E. S. Tucker); Labette, Marion and Montgomery Cos. (Biol. Survey); Norton Co. (C. D. Bunker); Osage Co. (Biol. Survey); Reno Co. (G. D. Hanna); Riley, Rooks and

Shawnee Cos. (Biol. Survey); Sumner Co. (writer); Trego Co. (Biol. Survey). It is also reported from Barber county by Cragin, under the name *Neolithobius mordax* (a synonym), and Bollman states that *L. mordax* occurs in Kansas.

20. **Poabius bilabiatus** (Wood).

Lithobius bilabiatus Wood. Proc. Phila. Acad., 1867, p. 130.
Poabius bilabiatus Chamberlin. Ann. Entom. Soc. Amer., vol. V, p. 153 (1912).

All of the specimens in the collection are of small size, varying from 7 mm. to 12 mm. in length. While this species is scarce, it seems to have a wide distribution in the state.

HABITAT: Cowley and Douglas Cos. (writer); Graham Co. (Biol. Survey); Sumner Co. (W. O. Riley); Trego Co. (Biol. Survey).

ORDER EPIMORPHA.

FAMILY CRYPTOPIDÆ.

21. **Otocryptops sexspinosus** (Say).

Cryptops sexspinosus Say. Jour. Phila. Acad., vol. II, p. 112 (1821).
Otocryptops sexspinosus Pocock. Ann. and Mag. Nat. Hist., ser. 6, vol. XV, p. 351 (1895).

Color, adult, deep orange; young, yellow. The largest specimens in the collection measure 57 mm. in length. This form is common under stones and boards, especially around old buildings. One specimen collected by the writer was carrying a mass of eggs suspended under the middle of her body, and held there by the legs. This mass measures 10 mm. long by 6 mm. wide. The eggs are yellow in color and measure 1.3 mm. in diameter.

Wood shows the anterior margin of the labium in this species to be nearly straight; in all of the forms studied, a median notch is well marked.

HABITAT: Anderson Co. (E. H. Taylor); Bourbon Co. (Biol. Survey); Butler Co. (E. C. Harrah and Biol. Survey); Cherokee Co. (Biol. Survey); Cowley Co. (writer); Crawford Co. (Biol. Survey); Douglas Co. (G. D. Hanna, Biol. Survey, and writer); Franklin, Graham and Greenwood Cos. (Biol. Survey); Jefferson Co. (writer); Labette Co. (R. D. Lindsey and Biol. Survey); Linn Co. (Biol. Survey); Montgomery Co. (R. D. Lindsey and Biol. Survey); Osage, Osborne, Pottawatomie, Riley, Rooks and Shawnee Cos. (Biol. Survey); Sumner Co. (writer); Trego Co. (Biol. Survey). It was previously reported from Jefferson Co. by Cragin.

FAMILY SCOLOPENDRIDÆ.

22. **Scolopendra heros** (Girard).

Scolopendra heros Girard. In Marcy's Exp. Red River, App. F, p. 262 (1853).

The largest specimen in the collection measures 114 mm. in length. So far found only in the southern part of the state.

Through the kindness of Prof. C. H. Edmondson, of Washburn College, I have examined two individuals of this species collected by Cragin in Barber county. The larger one measures 140 mm. These two specimens are all that remain of Prof. Cragin's Myriapoda collection, as the remainder were destroyed by fire.

HABITAT: Cowley Co. (writer); Elk Co. (Biol. Survey); Labette Co. (R. D. Lindsey and Biol. Survey); Montgomery Co. (Biol. Survey). This is also reported from Barber county by Cragin, as above mentioned.

23. **Scolopendra morsitans** (Linnæus).

Scolopendra morsitans Linnæus. Syst. Nat., X, p. 638 (1758).

Under the name *Scolopendra morsitans* var. *cærulescens*, Cragin reports a single specimen of this species from Barber county. It is a very doubtful record for the state.

24. **Scolopendra polymorpha** (Wood).

Scolopendra polymorpha Wood. Proc. Phila. Acad., 1861, p. 11.

Probably found throughout the state, but more common in the southern and western portions.

HABITAT: Butler and Chase Cos. (Biol. Survey); Cowley and Douglas Cos. (writer); Ellsworth Co. (A. Wellington); Graham and Greenwood Cos. (Biol. Survey); Jefferson Co. (writer); Labette Co. (R. D. Lindsey and Biol. Survey); Marion Co. (Biol. Survey); Montgomery Co. (R. D. Lindsey and Biol. Survey); Norton Co. (C. D. Bunker); Riley Co. (Biol. Survey); Russell Co. (W. S. Sutton); Trego Co. (Biol. Survey). Wood reports this species from Kansas, and it is recorded by Cragin from Barber, Finney and Rice counties.

FAMILY GEOPHILIDÆ.

25. **Arenophilus bipuncticeps** (Wood).

Geophilus bipuncticeps Wood. Jour. Phila. Acad., vol. V, p. 45 (1862).

Arenophilus bipuncticeps Chamberlin. Bull. Mus. Comp. Zoöl. Harvard, vol. LIV, p. 417 (1912).

This is one of the most common centipedes. It lives under dung, stones, and pieces of wood, and is often found in the loose soil of gardens. It seems to be able to stand dry weather better than most species, as they seem to be as numerous after a long dry spell as when the ground is moist.

HABITAT: Anderson Co. (E. H. Taylor); Bourbon Co. (Biol. Survey); Butler Co. (E. C. Harrah); Cherokee Co. (Biol. Survey); Cowley Co. (writer); Douglas Co. (E. H. Taylor, Biol. Survey, and writer); Graham and Greenwood Cos. (Biol. Survey); Jefferson Co. (writer); Labette Co. (R. D. Lindsey and Biol. Survey); Marion, Montgomery, Osage, Osborne, Pottawatomie, Riley, Rooks and Shawnee Cos. (Biol. Survey); Sumner Co. (W. O. Riley and writer); Trego Co. (Biol. Survey). Also reported from Jefferson county by Cragin.

26. **Arenophilus osborni**, n. sp. Plate XVI, figs. 1, 2 and 3.

Body robust, round, attenuated at both anterior and posterior ends. Color, Naples yellow, with a light brown, rather large spot on nearly every segment. Head and antennæ light orange; feet and sides lighter than dorsal; ventral side slightly lighter than dorsal, without dark spots.

Cephalic plate about as broad as long (32:34). The two sides of the anterior margin slightly concave, each extending from the center at a slight angle backwards. Sides somewhat convex, having well-rounded corners at the posterior ends. Posterior margin truncate, straight. Covered with fine punctations; also larger ones scattered over the surface, in each of which a hair probably grows, as several still contain them, making the cephalic plate sparsely pilose; hairs not arranged in rows, but irregularly scattered over the surface. A group of four or five hairs on lateral margins, just back of the anteriolateral corners.

Prebasal plate slightly exposed at the center.

Mandibles large, when closed reaching to the middle of the first joint of the antennæ. Armed at the base with a single very minute tooth. Slightly pilose.

Labium longer than broad (5:4). Very slightly emarginate. On type no trace of a median canal; on male (?) specimen median canal distinct, but this is probably due to the latter specimen having been dried before preserving. Finely punctate.

Antennæ rather long, the first five or six joints covered with long hairs, the remaining with short hairs. The hairs gradually become more numerous and shorter from the proximal to the distal end.

The dorsal plates are minutely punctate (similar to the head); bisulcate, with rather broad, shallow furrows; a few scattered hairs over the surface.

The ventral plates are distinctly bisulcate, with a small, nearly round median depression, which becomes somewhat elongate posteriorly.

The spiracles are oval on the first four or five segments, the remainder being round. They gradually become smaller towards the caudal end.

The coxal pores empty into two large pores on each side of a rather broad ventral plate (a generic character).

The anal legs are sparsely pilose and armed with a single claw. Not swollen.

Number of legs; ♂ (?), 53; ♀, 59. Length, ♀, 57 mm.

HABITAT: Cowley Co. (type) (writer); Bourbon Co. (Biol. Survey).

Named in honor of Prof. Henry L. Osborn, of Hamline University, my former teacher.

There are only two specimens of this species in the collection, and the one from Bourbon county is in bad condition owing to its having been crushed and dried before it was preserved. It has lost the anal legs, and so I am not positive it is a male, but believe so from the number of legs, 53, instead of 59, as in the female type.

27. **Geophilus dolichocephalus**, n. sp. Plate XX, figs. 4, 5 and 6.

Color, light orange, with a dark orange head. Young specimens are yellow with an orange head. Legs yellow. About the same size for the first two-thirds of its length, attenuated from this point backwards.

Head large, much longer than wide (5.5:3.4 or 6.3:4), with the cephalic margin slightly truncated, about equal in width to the caudal margin, which is distinctly rounded. The mandibles are large, extending to the base of the third joint of the antennæ. They are armed with four teeth, the distal one of which is prominent and black in the adult; the other three are small. The coxæ of the mandibles are longer than broad (19:11) and have a slight indentation on the inner surface, distally. The labium is large and square (1:1), and is clearly but not prominently notched. It is not canalate.

The antennæ are short, the first eight joints being sparsely covered with rather long hairs, and the distal seven joints being densely pilose with very short hairs.

The dorsal plates are distinctly bisulcate, with the exception of the caudal four or five.

The ventral plates have a distinct median depression, which anteriorly extends from the cephalic to the caudal margin of the plates, but posteriorly it becomes shorter, occupying the middle part as a long, oval depression.

The spiracles are oval cephalad and round mesad and caudad, growing smaller towards the posterior end.

The coxæ and joints of the anal legs are rather swollen. The coxal pores are round, arrangement irregular, except that most of them are on the ventral surface near and partly concealed by the ventral plate, while a few are grouped near the edge or partly under the dorsal plate. They vary in number from 21 to 25. The anal legs in the female are pilose with rather long hairs, while in the male they are covered with fine hairs, especially distally, among which are numerous longer hairs.

Legs, pilose; in number, 63 to 69 pairs. Length, 42-45 mm.

HABITAT: Bourbon Co. (Biol. Survey); Cowley Co. (type) (writer).

ETYMOLOGY: δολιχός, long; κεφαλή, head.

This species is related to *G. strigosus* (McNeill), recorded from Indiana, but differs from the latter in the following points:

G. strigosus (McNeill).	*G. dolichocephalus*, n. sp.
Head, longer than wide (3.3:2.1), cephalic margin truncate, caudal margin rounded.	Head, longer than wide (5.5:3.4 or 6.3:4); shape the same.
Antennæ short, sparsely pilose, almost bare proximal.	Antennæ apparently shorter; first eight joints sparsely covered with long hairs; distal seven joints densely pilose with very short hairs.
Mandibles armed with two very minute teeth each.	Mandibles large, armed with four teeth, the distal one large and black in adult.
Labium obsoletely canaliculate, and scarcely emarginate.	Labium not canaliculate, and distinctly emarginate, especially in adults.
Ventral plates with a distinct mesal depression, elongate, forming a shallow oval.	Well-marked mesal depression present; cephalad, forming a distinct groove from anterior to posterior margins of plate; caudad, forming an elongated oval.
Number of pairs of legs, 55.	Number of pairs of legs, 63 to 69. A specimen 22 mm. long has 69 pairs.
Length, 23.5 mm.	Length, adult, 42-45 mm.

The above characters applied to *G. dolichocephalus* are as true of specimens 25 mm. long as of the full-grown specimens.

I know little of the habits of this species, except that they were collected with and placed in a bottle containing a large number of *Arenophilus bipuncticeps* (Wood); so I judge their habits are similar.

28. **Geophilus mordax** (Meinert).

Geophilus mordax Meinert. Proc. Amer. Phil. Soc., vol. XXI, p. 217 (1885).

Rare. Those from Cowley county were collected under stones along the banks of a small creek in dry weather.

HABITAT: Cowley Co. (writer); Douglas Co. (G. D. Hanna).

29. **Linotænia fulva** (Sager).

Strigamia fulva Sager. Proc. Phila. Acad., vol. VIII, p. 109 (1856).

Linotænia fulva Bollman. Proc. U. S. Natl. Mus., vol. XI, p. 341 (1888).

Not common.

HABITAT: Cowley and Douglas Cos. (writer); Franklin Co. (Biol. Survey); Norton Co. (C. D. Bunker); Riley Co. (Biol. Survey).

KEY TO THE MYRIAPODA OF KANSAS.

The following key covers all of the species of Diplopoda and Chilopoda that have so far been recorded from the state of Kansas, with the exception of *Tylobolus uncigerus*, which undoubtedly does not occur in this state. In choosing characters to be used, those have been taken which seem to the writer to be the most readily understood, and to require the least possible study for their use, thus making as serviceable a key as possible. It is of course understood that this key has been made especially for the species enumerated, and many of the characters would not hold good if other forms were added to the list. In making the key, the following references have been of special value:

Bollman, Chas. H., "Myriapoda of North America," Bull. No. 46, U. S. National Museum (1893).

Chamberlin, Ralph V., numerous papers.

Cook, O. F., "Myriapoda of Northwestern North America," Harriman Alaska Exped., vol. VIII (1904).

KEY.

* Most of the segments with two pairs of legs; antennæ with 5 to 8 segments: one pair of maxillæ (DIPLOPODA).

A. Segments 20; repugnatorial pores on 5, 7, 9, 10, 12, 13, 15-19 segments.

B. Femora of legs spined; lateral carinæ more or less bent downwards. *Fontaria virginiensis.*

BB. Femora of legs not spined.

C. Dorsal plates smooth with a longitudinal sulcus; lateral carinæ small; body narrow, lives in hothouses . . . . . *Oxidus gracilis.*

CC Dorsal plates smooth or scaly, not distinctly sulcate; lateral carinæ large; body wide.

D. Back convex, smooth; lateral carinæ not serrate.

E. Male genital appendages short, their terminal spine single, curved, densely pilose. . . . *Leptodesmus hispidipes.*

EE. Male genital appendages elongate; terminal spine large, spiral, furnished with a long, falciform, basal spinule, and a second short, robust lateral one.
Leptodesmus placidus.

DD. Back flat, scaly; lateral carinæ serrate.

E. Repugnatorial pores surrounded by a round swelling; scales on scuta arranged in 4 transverse rows.
Scytonotus granulatus.

EE. Repugnatorial pores surrounded by a long, oval swelling; scales arranged in 3 transverse rows.

F. Length from 15 to 18.5 mm., *Polydesmus pinetorum.*

FF. Length from 22 to 27 mm.... *Polydesmus serratus.*

AA. Segments 30; repugnatorial pores absent; anal segment produced into two slender papillæ; dorsal face of each segment provided with 6 bristles; carinæ reduced.
Cleidogona sp?

AAA. Segments more than 30; repugnatorial pores present.

B. Male genital appendage formed from anterior pair of legs of 7th segment; sides and back with numerous keels; anal segment not produced into spine ..... *Lysiopetalum lactarium.*

BB. Male genital appendage formed from both pairs of legs of 7th segment; no carinæ or keels; anal segment produced into spine.

C. None of anterior segments apodous..... *Actobolus marginatus.*

CC. Third segment apodous; male genital appendages composed of two parts each.

D. Inner part of male appendage curving towards opposite side and crossing with opposite similar part; outer part ending in one blunt point.

E. Outer part of male appendage narrow in proportion to length (9:34) ..... *Parajulus venustus.*

EE. Outer portion of male appendage wider (16:34)
Parajulus impressus.

DD. Inner part of male appendage curving caudad, meeting in a mid line, but not crossing as above; outer part ending in two blunt points ..... *Parajulus diversifrons.*

** Not more than one pair of feet to the segment; antennæ many segmented; two pairs of maxillæ (CHILOPODA).

A. Pairs of legs, 15; young born with 7 pairs.

B. Tarsi of legs multiarticulate; dorsal plates, 8; spiracles, 6, in one series, middorsal ..... *Scutigera forceps.*

BB. Tarsi of legs triarticulate, dorsal plates, 15, spiracles in two series, lateral.

C. Inside of anal legs with strongly marked, irregular swellings.
Poabius bilabiatus.

CC. Anal legs not as above.

D. Angles of 7th, 9th, 11th and 13th dorsal plates produced; prosternal teeth, 10 to 14 ..... *Lithobius transmarinus.*

DD. Angles of 9th, 11th and 13th dorsal plates produced; prosternal teeth, 8 to 1.. *Lithobius forficatus*

DDD. Angles of the 11th and 13th (or 13th only) dorsal plates produced; prosternal teeth, 4 ..... *Lithobius kansasensis.*

DDDD. Angles of none of dorsal plates produced; prosternal teeth, 4 to 6 ..... *Lithobius iowensis*

AA. Pairs of legs, 21, young born with same number, ocelli, 4.

B. First dorsal plate with a deep transverse furrow back of anterior margin

C. Cephalic plate with two fine longitudinal lines, which diverge cephalad
Scolopendra heros.

CC. Cephalic plate without furrows as above ..... *Scolopendra polymorpha*

BB. First dorsal plate without a deep transverse furrow back of anterior margin
Scolopendra morsitans

AAA. Pairs of legs, 23, young born with same number, ocelli absent.
Otocryptops sexspinosus

AAAA. Pairs of legs more than 30; young born with same number.
B. Dorsal plates bisulcate; middle piece of labrum small.
C. Coxal pores empty into two large pores on each side of a broad ventral plate
D Cephalic plate clearly longer than broad, with a distinct line of deep punctations each side of middle on caudal half, these connected by a cross line of punctations in front.
Arenophilus bipuncticeps.
DD. Cephalic plate almost as broad as it is long, with punctations scattered irregularly over surface.....*Arenophilus osborni.*
CC. Coxal pores numerous, not as above.
D. Head only slightly longer than broad (almost round); 49 to 55 pairs of legs.....*Geophilus mordax.*
DD. Head distinctly longer than broad (oblong); 63 to 69 pairs of legs .....*Geophilus dolichocephalus.*
BB. Dorsal plates not bisulcate, middle piece of labrum very large.
Linotænia fulva.

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THE

KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 7—January, 1913.

(Whole Series, Vol. XVII, No. 7.)

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NOTICE OF A NEW FISH FROM THE PERMIAN OF KANSAS, WITH DESCRIPTION, ... *H. T. Martin.*

PUBLISHED BY THE UNIVERSITY

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 7] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No. 7

NOTICE OF A NEW FISH FROM THE PERMIAN OF KANSAS, WITH DESCRIPTION.

BY H. T. MARTIN.

Ctenoptychius semicircularis Newberry and Worthen.

(Contribution from the Zoölogical Laboratory, No. 206.)

A SINGLE TOOTH referable to this species was discovered about a year ago by Mr. J. W. Glydden, of Lawrence, while working on a block of stone shipped here from the quarries situated about thirty miles east of Arkansas City, Kan. The formation from which the block was secured is evidently Permian. When received the tooth was firmly embedded in a mass of limestone. By careful work the specimen has been entirely cleaned from the adhering matrix, and now presents a nearly perfect tooth.

Like similar teeth described by Newberry (Geological Survey of Ohio, vol. 2, page 41), this specimen has a covering of enamel on the upper or dentigerous portion, of a light gray color; the enamel has a clearly defined line of demarcation on the outer or convex side, which extends 9 mm. down from the tip of the large center denticle, while the whole inner or concave cup of the tooth is entirely covered with enamel, which overlaps the inner wall or heel of the tooth, and extends 4 mm. down the outside of this inner edge. The tooth has seven saw-tooth-like denticles, forming the higher semicircle and what is probably the outer edge, the center denticle much the largest. The enamel surface is very smooth and polished. The root or base of the tooth is of a brown color, and is lightly scored all around with vertical lines. A cross section of the base shows

Received for publication September 3, 1912.

numerous small tubular openings for the nutriment of the tooth to pass through. As a whole, the base resembles that of the common fossil shark *Lamna* or of *Ptychodus* in general structure. As far as the writer is aware, this is the first time that *Ctenophychius* has been reported from the Permian of Kansas. The following measurements have been made:

	mm.
Width of tooth	16
Depth from middle denticle to base	20
Length of longest denticle	4
Width of longest denticle at base	5
Depth of inner cup from point of longest denticle	14

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 8 — June, 1913.

(Whole Series, Vol. XVII, No. 8.)

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HISTOLOGY OF *JANUSIA GRACILIS.*

BY B. J. CLAWSON.

Plates XXII XXVIII

(Contribution from the Botanical Laboratory.)

JANUSIA GRACILIS was one of the plants collected by the University of Kansas from the desert region of Arizona by Mr. Peace. It was collected in the year 1908 in the vicinity of Tucson, Ariz., from Tumamoc Hill, on which the desert laboratory stands. There are four topographical divisions found in this desert region: mesa land, river bottoms, rocky slopes of mountains, and draws in these mountains. Of these four the rocky slopes of the mountains are the dryest. It seems that this plant is one of those best fitted for the dry, hot conditions, since it survives upon these slopes where the soil is just thick enough to cover the rocks, which are broken so that what moisture the soil could collect from the rain is soon drained away.

A short description of the genus *Janusia* is given by Bentham and Hooker. Gray mentions both the genus and species in his catalogue of North American Plants. Not even the family to which the genus belongs is mentioned in the seventh edition of his manual. Engler and Prantl give a description of the genus similar to that given by Bentham and Hooker. They also mention the species, saying that it is found in Mexico.

The material studied was an average of that found upon the field, and was a slender, twining, woody shrub. In no place had it reached the thickness of one centimeter.

Received for publication October 8, 1912.

Solereder describes the anatomical features of the family to which the plant belongs, the Malpighiaceæ. It seems to be an inconspicuous family. He mentions three characteristics of the family, which are the following: No internal secretory cells; the subsidiary cells of the stoma run parallel to the pores of the stoma; and the unicellular trichomes. While all of these were noticed in our *Janusia*, the last seemed to be the most important, as it appears to take part in collecting of water, as will later be shown. It would not seem at all unreasonable that if a plant had to modify any of its parts to adapt itself to its environment, it would modify those parts most characteristic of it. In this plant that is what has evidently taken place, for, as we shall see, the trichome is not a typical Malpighian hair, which is so characteristic of the family.

The root was not worked, as none of it was collected. The stem and leaves were collected in July, just after the rainy season had begun, which extends over July and August. The leaves were well developed, but not old. It would be of interest to study the leaf in the bud to get the plan of the development of the trichome. This was not done, as proper material was not to be had at the time when the plant was worked. No description of the species has been found in the literature to which the writer has had access. The literature cited will be general in character.

METHOD OF PROCEDURE.

The work was begun by cutting free-hand sections and staining with c. z. i. and phloroglucin. By these means it was determined concerning the parts lignified, cutinized, and suberized.

Hydrochloric and acetic acids were used for the determination of the chemical composition of the crystals found in the phloëm and in the leaf. Chloride of iron was used for the determination of the tannins, which were found in great abundance in both the stem and leaves.

The stems were softened by soaking them in concentrated hydrofluoric acid for ten weeks. A description of this method is found in the *Botanical Gazette* (vol. 37, year 1904, by Amon B. Plowman). He recommends the use of a 10 per cent solution of the hydrofluoric acid. In this work the concentrated solution seemed to do the work and leave the tissue of the plant in perfect condition.

Material was mascerated with nitric acid and potassium chlorate, and also with chromic acid. The former proved superior for the wood fibers and the wood parenchyma, but the latter gave better results for the bast fibers.

All sections, except those cut free-hand with the razor, were cut on the rotary microtome, the material being embedded in paraffin. Bismark brown was the main stain used, though hæmatoxylin, alcohol safranin, and chloral hydrate safranin were used.

The leaves were bleached by a method worked out in this laboratory by Mr. Peace. An account of this method is given in the *Plant World*, April, 1910. Good results were obtained in getting the venation by staining these bleached leaves with chloral hydrate safranin. It was also found to be an advantage to use a second stain, Bismark brown, for the study of the epidermis and stomata. In making detailed study of the pits in the stem, dilute glycerine was used for a mounting fluid, but permanent mounts were made in Canada balsam.

THE STEM.

The pith in the young stem occupies about one-fourth of the diameter of the entire stem (fig. 1). In the older it becomes much smaller (fig. 2). In the young stem the pith is made up of thin-wall cells, elongated vertically, the wall of which gives the cellulose reaction with c. z. i. The marginal pith cells vary in size, averaging .015 mm. in breadth, and from .074 mm. to .185 mm. in length; while the mid-pith has larger cells, which are not so regular in shape. These average about .03 mm. to .045 mm. in diameter, being in length about the same as those of the margin.

In the marginal pith cells is found a granular, reddish-brown substance which is insoluble in water, chloroform, alcohol, xylol, chloral hydrate, hydrochloric acid, and hydrofluoric acid. It is not affected by any of the stains used in this work. With ferric chlorid it reacts and gives a black color. From this reaction it seems that it is a tannin of some kind. It is not a pure tannin, as this is soluble in alcohol at 85 sp. g. This material was passed through alcohol from 20 per cent to 98 per cent. Its reaction with ferric chlorid, when compared with the reaction of tannin in *Sambucus*, was found to be identical. In the older stem the pith cell walls become much thicker, and the cells become longer. They are filled with

plain pits, which give the appearance of stone cells (fig. 21). The xylem occupies about two-fifths of the entire stem (figs. 1, 2 and 11). It is made up of wood parenchyma, wood fibers, fiber tracheids, and tracheal tubes. Secondary thickening of the cell walls has taken place to a large extent. The cells in their growth have become crowded, and there is very little intercellular space to be found, and this decreases as the xylem advances toward the phloëm. No distinct rings of growth occur. This seems to be characteristic of many of the desert plants, which may be the result of the short growing season. The cells of the xylem are so similar in appearance in cross section that medullary rays are distinguished with difficulty even in young stems, but they may be seen in tangential and radial longitudinal sections (fig. 11). They are made up of from one to three rows of cells tangentially. According to Solereder, this is a characteristic of the anatomy of the family Malpighiaceæ. As seen in radial longitudinal section, they are thick-walled cells, elongated vertically, and have many plain pits (fig. 19).

The cells of the xylem give the lignified reaction with c. z. i. and phloroglucin. They are exceedingly hard because of the deposit of silica in the cell wall. It was with difficulty that xylem tissue could be cut with the razor. After soaking it for ten weeks in hydrofluoric acid and washing the acid out in running water, both longitudinal and cross sections were easily cut with the rotary microtome from material embedded in paraffin.

The water-conducting system, seen in the stem cross section, is conspicuous. Considering this system from the standpoint of the percentage of the total area of the stem taken up by tissue devoted to the function of carrying water, it could be considered fairly well developed. The tracheal tubes average from .01 mm. to .052 mm. in diameter. The number of tubes counted in a cross section of a stem of 1.6 mm. in diameter was 260, and we find that about 10 per cent of the stem's cross section is occupied by tracheal tubes. Near the pith the spiral tubes are found, while toward the phloëm the reticulated tubes are found. These tubes are well supplied with pits, all of which are bordered (figs. 14 and 18).

Besides the tracheal tubes, many fiber tracheids are also found. These occur scattered through the xylem, intermixed with wood fibers and wood parenchyma. They resemble the

wood fibers, except that they have bordered pits, while the wood fibers have plain pits. In this plant this is the only distinguishing feature between the wood fibers and the fiber tracheids. These fiber tracheids vary in length from .07 mm. to .195 mm. In width they are from .01 mm. to .018 mm. They exceed the wood fibers in number to a considerable extent, but the exact proportion was not determined. Scattered among the tracheids are wood fibers and wood parenchyma. The wood fibers average about .255 mm. in length and .01 mm. in width, having walls of the thickness of .003 mm., in which there are many plain pits (figs. 3, 4, 5, 6 and 7).

The wood parenchyma cells average from .22 mm. to .33 mm. in length and .03 mm. in width. These also have plain pits in their secondary thickening (fig. 14).

A well-defined cambium is not found in *Janusia,* but the differentiation between the xylem and the phloëm is distinct (fig. 11). The phloëm occupies a relatively small amount of both the young and old stems (figs. 1, 7 and 2). It forms a continuous ring, except where it is broken by the medullary rays, and has an irregular outline. The cells are small and elongated vertically, with very little if any other differentiation, and in these calcium oxalate occurs. Their walls are cellulose, and so far as could be determined there were not any sieve tubes. Sections ranging in thickness from seven microns to fifteen microns were examined with different stains, and with the two-thirds, one-sixth, and one-sixteenth objectives, and no sieve tubes were found.

The arrangement of the pericycle is similar to that of the usual stem. In young stems it is made up of parenchyma cells and bast fibers. The bast fibers are arranged in a discontinuous ring. They vary in length from 1.5 mm. to 2.5 mm., and in width from .007 mm. to .03 mm., and have plain pits. In the old stem the pericycle has differentiated palisade cells next to the endodermis, and in these palisade cells chloroplasts are found (fig. 22). In the inner part of the pericycle are the ordinary parenchyma cells and bast fibers.

The endodermis is well defined and is filled with starch. Outside of the endodermis, clear to the epidermis, are thin-walled parenchyma cells, no collenchyma being present. The epidermis is made up of one row of cells, in which are stomata and pores (fig. 12). Over these pores are unicellular trichomes. (A discussion of these will be given under the description of

the leaves.) The number of the stomata in the stem averages 67 per sq. mm., the pores being somewhat less in number; but the stomata have subsidiary cells parallel to the pores of the stoma, which is another special characteristic of the family, as given by Solereder (figs. 17 and 23). The cell walls of the very young stems are composed of cellulose, while in the older stems all of the tissue exterior to the endodermis has become suberized, thus forming cork. In the old stems no remains of the epidermis appear. It may be that the cork has been formed by the cork cambium, though no definite layer is found (fig. 15). It would seem from the location of the cork that if there were a cambium layer it has arisen from the endodermis.

THE LEAF.

The leaf is oblong, elliptical in form, tapering to a flat base (fig. 25). It is from 8 mm. to 1 cm. long, about 3 mm. wide, and about .13 mm. thick.

It is covered with hair on both upper and lower surfaces. These hairs are placed upon the leaf so as to run longitudinally with the leaf (fig. 26). In length they average about .9 mm. and have a diameter of about .025 mm. They taper to a point at both ends, and their surface is covered with papilla-like projections (fig. 29). These hairs are free at both ends and are fastened to the leaf by a stalk-like projection which fits into an opening of the epidermis. This opening resembles a pore (figs. 28 and 29). This kind of a hair is an anatomical characteristic of the family Malpighiaceæ according to Solereder, in his Comparative Anatomy of Dicotyledons. The leaf contains about 67 of these pores per sq. mm. on the upper surface, and about a similar number on the lower surface. The arrangement of the trichome is a peculiar one. Nothing quite like it has been described as far as is known by the writer. Trichomes similar in shape are of common occurrence—especially is this true in the family Malpighiaceæ—but none of them has become differentiated to such an extent as is found in the trichome of *Janusia*. It is uncommon to find a hair on the leaf with the walls cellulose throughout. Solereder describes a hair having a cellulose membrane around it. The hair of *Janusia* differs from this in that the walls of its hairs are all cellulose except at the base of the stalk.

Many examples of trichomes are found whose structure is very similar to that of *Janusia*. Examples of Ranunculaceæ

are found where the trichome rests on a stalk and is fastened to the epidermis at a point where the cells which surround this stalk are raised above the surface of the leaf.

It may be that this hair has something to do with the absorption of water, which has to be used with so great economy by the desert plants. Haberlandt describes a hair of *Diplotaxis harra* that is used for such a purpose. It is cutinized, and at the base has pits through which water may pass from the surface of the hair to the inner part of the hair, and then to the epidermal cells and on to the mesophyll. Since the two-armed trichome is so characteristic of the Malpighiaceæ—so characteristic that they are called Malpighian hairs—it would not seem at all improbable that the plant would modify one of its most common structures in adapting itself for desert conditions. It may be possible that during the night, when the hair would give off heat enough by radiation to bring the temperature of the air to the point of saturation, water would be collected into the cellulose hair; from this the water would pass by filtration or osmosis into the cavity of the pore under the membrane through that part of the stalk of the trichome which has cellulose walls (fig. 29, *C*). As the cavity would fill up, the membrane would be forced outward, lifting with it the hair, thus opening the pore beneath. The plug would fall into position again as soon as transpiration reduced the pressure from below the membrane. The cutinized end of the plug would then retard further loss of water at this point. This explanation of the function of the hair, at its best, is only theoretical. No fresh material was at hand for physiological experiments. Several difficulties arise in applying this theory of the function of the hair. Would water pass through a cellulose cell wall and fill the cell cavity if the cell were dead? Whether the cell contained protoplasm at the time of collection was not determined. It would be easy enough for the water to pass from the outside of the hair to its inside, provided the cell was living and filled with protoplasm, but the question then arises, even if the cell were alive, how could it live in the desert with only cellulose walls to keep it from drying up? It would be interesting for one having fresh material to work out the ontogeny and physiology of these hairs.

In the Dilleniaceæ simple unicellular and multicellular trichomes may be seen loose at each end and fastened at the

middle by a stalk-like arrangement. The stalks are usually embedded in the epidermis. The hair is raised upon a pedestal and the cuticle of the epidermis is continued on up around the pedestal. This is similar to that found in *Janusia*. In Calycanthaceæ unicellular trichomes are found having two arms and a stalk. This stalk fits into the epidermis and is surrounded by a rosette-like arrangement of the epidermal cells. This rosette arrangement is like that found surrounding the stalk of *Janusia*. The Cruciferæ furnish examples where the stalk of the trichome extends down below the epidermal cell layer. In the Pittosporeæ the hairs resemble those of *Janusia*, except that the stalk is multicellular, while that of *Janusia* is but part of the whole hair. Hairs with papillæ are not of an uncommon occurrence. This is a characteristic of the hairs of *Janusia*.

In figure 29 is shown a cross section of the leaf and hair in position. Its stalk tapers to a round point (fig. 29, *A*). The point is cutinized, but the upper portion of the stalk, including the rest of the entire hair, is cellulose. In cross and longitudinal sections the hair seems to be hollow.

It is held in place by the continuation of the outer walls of the epidermal cells, which is cutinized as far up as to the attachment of the hair (fig. 29, *B*). This membrane holding the hair in place fastens to the cell surrounding the pore at about the middle of their upper surface. These cells surrounding the pore have walls with a thicker cutinized substance than the adjacent cell walls.

As is seen, the palisade cells extend up directly against the pore. It is evident that the trichome has developed from one cell and that it has become broken loose from those cells surrounding it, thus leaving it loose to move up and down. In this way it may form a plug in the pore at times and leave it open at other times. Whether the protoplasm has been disorganized has not been determined. At different times the hair has been found collapsed.

The epidermal cells have a depth of about .018 mm., and have a thick outer cutinized wall. The palisade cells are very compact and leave but little intercellular space. In length they are about the average, but in width they are very narrow, having a width of about .015 mm. (figs. 24 and 29). The chloroplasts are very small and very numerous. These pal-

isade cells are only found in the upper surface, except over the mid rib, where large cells resembling the border parenchyma are found (figs. 24 and 33). In most of these there is an abundance of tannin. while others seem to be used as water-storage cells.

The spongy parenchyma is similiar to that found in the usual leaf. The lower epidermis has cells somewhat smaller than the upper epidermis, and in it are found both stomata and pores (fig. 25). The number of free endings of the veins is about 100 per square mm., having an average distance between them of .011 mm. The veins in the mesh work average in distance apart from .2 mm. to .9 mm. Having the endings so close together would seem to indicate that a great deal of water might pass through them. So far as this anatomical detail is concerned, the leaf is equipped to supply rapid transpiration of water.

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THE

KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 9 — June, 1913.

(Whole Series, Vol. XVII. No. 9.)

CONTENTS:

SOME SHELLED RHIZOPODS OF KANSAS—A PRELIMINARY REPORT ..... *Regina Woodruff.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post office in Lawrence as second class matter

1-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 9] JUNE, 1913. [WHOLE SERIES VOL. XVII, No. 9

SOME SHELLED RHIZOPODA OF KANSAS.

A PRELIMINARY REPORT.

BY REGINA WOODRUFF.

Plates XXIX XXXVI.

(Contribution from the Zoological Laboratory, No. 207.)

THE classification of the American Protozoa has received little attention until the last few years. Stokes' "A Preliminary Contribution toward a History of the Freshwater Infusoria of the United States" and Leidy's "Freshwater Rhizopods of North America" are the only general works on the subject. In 1905 Conn, in connection with the Geological Survey work, made a preliminary report on the "Protozoa of the Fresh Waters of Connecticut," and in 1906 Edmondson's "Protozoa of Iowa" made its appearance. There is as yet no classification of the Protozoa of Kansas, though the advantage of such a work is evident.

It is the purpose of the zoölogy department of the University of Kansas to begin such a classification. To this end the following paper is contributed, and I want to take this opportunity to express my gratitude to Miss Nadine Nowlin, who directed the work.

CLASSIFICATION.

PHYLUM PROTOZOA.
- Class RHIZOPODA.
  - Order 1. Lobosa.
    - (*a*) Gymnamœbida (without shells).
    - (*b*) Thecamœbida (with shells).
  - Order 2. Heliozoa.
  - Order 3. Foraminifera.
  - Order 4. Radiolaria.

Received for publication October 8, 1912.

RHIZOPODA.

Pseudopodia of variable forms, blunt, sharp or anastomosing; naked, or with a shell which is either homogeneous or made of plates, sand grains, diatom shells, etc.

All of the following forms belong to the suborder Thecamœbida of the order Lobosa.

DIFFLUGIA Leclere.

Shell formed of foreign material, grains of sand, diatom shells, etc., united by a chitinous connecting substance.

Form spherical to pear-shaped, frequently with spine at the posterior end. Mouth occasionally excentric. Body not completely filling the shell, with finger-like pseudopodia. Nucleus and contractile vacuole variable.

Difflugia reproduce by an extrusion of a portion of the cell mass and by the formation of a new shell about the extruded portion, after which the two individuals separate.

Difflugia corona Wallich. Fig. 12.
Difflugia globulosa Duj. Figs. 7, 8, 13, 14, 16, 17, 20, 21, 29, 30, 32.
Difflugia lobosloma Leidy. Figs. 2, 4, 33.
Difflugia pyriformi Perty. Figs. 1, 2, 5, 6, 9, 10, 19, 22, 28, 31.
Difflugia arcula Leidy. Fig. 11.
Difflugia coristricta Ehrenberg. Figs. 15, 23, 24.
Difflugia acuminata Ehrenberg. Figs. 18, 26.
Difflugia urceolata Carter. Figs. 25, 27.

Difflugia corona Wallich.

Shell spheroidal, composed of particles of sand, the fundus usually spined. Mouth terminal, circular, with notched border. Is one of the larger species of the genus. Distribution general, found usually with algæ. The small tooth-like processes bordering the mouth may vary in number, but are usually more than ten. The number of spines also varies, but seldom have less than four or five to an individual been observed. Plate XXX; fig. 12.

Difflugia globulosa Dujardin.

Shell more or less spheroidal in shape, composed usually of quartz sand. Mouth large, terminal, seldom with a well-marked neck. One of the smallest and rarest species of the genus. Fundus usually evenly rounded, bearing no spines. Mouth is large, round and without lobes. The habitat of this species is similar to that of other members of the genus. Plates XXIX, XXX, XXXI, XXXII; figs. 7, 8, 13, 14, 16, 17, 20, 21, 29, 30, 32.

Difflugia lobosboma Leidy. .095 length.

Shell small, oval, usually composed of fine sand grains. Mouth terminal, small with 3 or 4 well marked lobes. Psd. as in other members of the genus. Common in pond water and has wide distribution. Plates XXIX, XXXII; figs. 2, 4, 33.

Difflugia pyriformis.

Shell oval or pear-shaped, sometimes with a short neck and broadly expanded fundus with or without spines, composed mainly of sand grains,

Psds. slender, cylindrical, simple or branched. Species is a common one and many varieties of shells are included in it. Plates XXIX, XXXI, XXXII; figs. 1, 2, 5, 6, 9, 10, 19, 22, 28, 31.

Difflugia arcula.

Shell hemispheroidal; fundus convex, base inverted shallow infundibuliform; mouth inferior, central, trilobed. Structure of shell usually of yellowish chitinoid membrane, usually with more or less adherent dirt or scattered particles of quartz sand or diatoms, especially occupying the fundus. Plate XXIX; fig. 11.

Difflugia acuminata.

Shell usually composed of sand crystals, oblong oval in shape, the fundus prolonged into an elongated process more or less acuminate; no spines. Great variations may occur in the size and shape of the shells. The posterior acuminate process, although a characteristic of this species, may be possessed by other species. Mouth large, terminal, circular. Plate XXXI, fig. 18; plate XXXII, fig. 26.

Difflugia constricta Leidy.

Shell oval in form with the anterior border obliquely truncate, composed of sand grains and other foreign materials. Fundus rounded, with or without spines. If present the number of spines may vary from one to half a dozen. Mouth large, circular or oval, and inverted. Shell colorless, yellowish or brown. Sarcode colorless. Plate XXX; fig. 15; plate XXXI, figs. 23, 24.

Difflugia urceolata.

Shell of large size, fundus usually evenly rounded, seldom spined; neck short; mouth large, circular, surrounded by a reflected rim with a thin edge. Pseudopods as in *pyriformis* rarely branched. (Length 250-300 microns.) Plate XXXI, fig. 25; plate XXXII, fig. 27.

Pseudodifflugia.

Shell chitinous, to which foreign materials are added. Pseudopodia delicate and branched. Sarcode mostly colorless, with large clear nucleus, but usually with sarcode obscured by the nature of the shell. Mouth terminal, inferior.

Pseudodifflugia gracilis.

Shell of variable form, mostly spheroidal, ovoidal, or oblong oval and of uniform transverse diameters, usually straight, occasionally slightly curved. Mouth terminal, circular. Structure of shell mostly obscurely granular, often of fine sand, frequently almost entirely composed of coarse sand, colorless or brownish. Found in ooze of ponds, ditches, etc. Plate XXXII, figs. 34, 35, 36.

ARCELLA.

Arcella Ehr.

Body secretes a shell of chitinoid material, semispheroidal in shape when viewed laterally, usually circular when viewed dorsally or ventrally. Mouth central. Psds. digitate or broadly lobate, never branched. Nuclei and C. Vac. usually more than one.

Arcella discoides.

Shell mostly circular, flattened disc-like, usually with height of from ¼ to ⅓ of the breadth; dome low, evenly convex to rounded or slightly expanded and rounded basal border. Base and mouth as in *A. vulgarus.* Color of shell yellow to dark brown. Plate XXXIII, figs. 42, 43.

Arcella vulgaris.

Shell pale yellow to dark brown; watch-glass shape, curved or polyhedral. Dome evenly convex or angularly faceted or concavely pitted at the summit and sides. The facets or pits variable in number. Mouth circular, mostly entire, rarely slightly crenulated at the border. Sarcode does not completely fill shell. Pseudopods digitate. Nuclei and vacuoles usually numerous. Common in ooze of almost all standing fresh waters. Lawrence golf links. *Arcella vulgaris* Ehrbg. Plate XXXIII, figs. 37-40, and 43-47; plate XXXIV, figs. 48-50.

Euglypha Duj.

Shell of round oblique rows of plates whose edges cross to make hexagonal areas. Shell spherical to pear-shaped. Spines frequently present. Mouth surrounded by regular teeth. Sarcode colorless, usually finely granular in posterior portion, coarsely granulated in anterior portion. Pseudopodia fine, anastamosing.

Euglypha alveolata Duj.

Shell composed of oval plates overlapping each other and arranged in regular rows, producing the appearance of hexagonal areas. Spines often present. *E. alveolata* is a common rhizopod, being found almost everywhere in fresh water among algæ or in the ooze at bottom of ponds. When the spines are present they are long, slender, and often curved, may vary in number from 4 to 6, and are variable in length. Plate XXXIV; figs. 56, 58.

Euglypha ciliata. Found in connection with figure 58.

Shell compressed ovoid. Mouth transversely oval. Fundus and lateral borders mostly fringed with spines which vary in number and size. Plate composing shell mostly elongated hexahedral, closely fitting at the margins. Mouth bordered by from 6 to 14 or more blunt, angular, crenulated teeth. Plate XXXIV; fig. 57.

Assulina Ehrenberg.

Shell compressed spherical or oval, neck almost obsolete, terminating in a transversely elliptical mouth with uneven or ragged edges, composed of minute oval or hexagonal plates in alternating rows. Sarcode and pseudopods as in *Euglypha.*

Assulina seminulum Leidy.

Shell nearly as broad as long, oval in shape, compressed, spineless, plates hexagonal or oval. In color ranges from dark brown to transparency. Plates are arranged in alternating rows and at the oval aperture give a notched appearance to the border. Pseudopods are delicate and transparent, branched but not anastomosing. Ooze at bottom of ponds. Plate XXXIV; fig. 59.

Trinema Duj.

Body enclosed within an elongated chitinous shell. Fundus rounded oval; extremity narrower and obliquity truncate. Mouth circular, sub-

terminal in center of the truncated border. Pseudopodia very delicate and transparent.

Trinema euchelys Leidy.

Usually the shell presents a homogeneous appearance, with a smooth outline, but in the larger specimens the shells show evidence of being made up of minute oval plates. The species varies greatly in size, as also in the density of the material composing the shell. Some are transparent, while others are completely opaque. *T. euchelys* is one of the smallest and most abundant of fresh-water rhizopods. Its distribution is wide. Plate XXXIV; figs. 51-55.

Centropyxis Stein.

Shell composed of chitinous material, with the addition of sand grains, diatom shells, etc., ovoid in shape, with the mouth and fundus eccentric in opposite directions. Spines usually present.

Centropyxis aculeata Stein.

Having characters of the genus *C.* may be considered as an intermediate form between *Arcella* and *Difflugia.* The color is usually some shade of brown; 1 to 6 spines are usually present, which are longer and more slender than those of *Dif.* and which seldom have any addition of foreign material. Plate XXXV; fig. 60.

Centrpyxis ecornis.

Shell composed of yellowish or brownish chitinoid membrane incorporated with quartz sand. Spineless mouth irregular in shape. Mouth and fundus eccentric in opposite directions. Plate XXXV; fig. 61.

Hyalosphenia Stein.

Shell compressed ovoid to pyriform, composed of transparent structureless chitinoid membrane; mouth terminal, inferior, transversely elliptical. Sarcode mass occupying the interior of the shell to a variable extent, attached to its inner surface by divergent threads and also connected to the border of the mouth. Protoplasm granular, mingled with colorless or colored corpuscles or both together. Nucleus large, contractile; vescicles several; pseudopods few, digitate.

Hyalosphenia papilio Leidy.

Has characters of genus. Shell composed of transparent, yellowish chitinoid membrane. Contains some green bodies. Pseudopods colorless and digitate. Found in mossy ooze from golf links, Lawrence, Kan. Plate XXXVI; fig. 64.

Hyalosphenia cuneata. Found in connection with 64.

Shell small, transparent, with encysted form. A large diatom shell extended up through the mouth. These forms were found in green mossy material collected from the gully on the golf links, Lawrence. Plate XXXV; fig. 63.

Pamphagus Bailey.

Body enclosed within a transparent hyaline, elastic membrane. Endoplasm completely filling the membrane; more or less granular. A large clear nucleus. Con. Vac. not distinctly determined; mouth small, terminal. Pseud. long, delicate, branching but not anastomosing.

Pamphagus mutabilis Bailey.

Body ovoid in shape, seldom changing form, the oval extremity more

acutely rounded than the fundus. Mouth small, often indicated only by the region from which the long, delicate pseudopodia are extended. Nucleus large, spherical. The closely investing covering of the body prevents any great distortions, although slight changes of form sometimes occur. As a result of stimuli the organism withdraws its pseudopodia and assumes a rounded or spherical form. *P.* no doubt represents a development of amœba to such an extent that the ectoplasm has become differentiated to form a resisting membrane. Plate XXXVI; figs. 65, 66, 67.

Pamphagus hyalinus.

Animal spheroidal and feebly produced at the lower pole into a short, broad neck, terminating in a circular mouth. Shell membranous, transparent, colorless, structureless and elastic. Sarcode intimately adherent to the interior of the shell, finely granular, with variable proportions of oil-like molecules. Nucleus large. Pseudopods filamentous, delicate and furcate. Plate XXXVI; fig. 68.

Nebella Leidy.

Shell usually compressed pyriform, transparent, colorless, with or without appendages, composed of cancellated membrane or of peculiar intrinsic structural elements of variable form and size, mostly of circular or oval disks, of narrow rectangular plates or rods, or of thin, less regular, angular plates, often almost exclusively of one or the other, sometimes of two or more intermingled in variable proportions. Sometimes of chitinoid membrane incorporated with more or less extrinsic elements, and sometimes of these entirely. Mouth inferior, terminal, oval. Sarcode colorless, in form, constitution and arrangement as in *Difflugia.*

Leidy records this form as being found only in sphagnum moss or in cedar swamps; therefore I hesitated about putting it in, but afterwards verified my finding in some mossy material that I know came from the gully on the golf links at Lawrence.

Nebella collaris Leidy.

Shell compressed pyriform, longer than broad, sides sloping downward and generally slightly inflected toward the oval end. Mouth transversely oval, entire. Shell colorless, variable in its structural elements, generally composed of oval or circular disks, sometimes nearly exclusively of one or the other, sometimes mingled with rod-like or narrow rectangular plates and sometimes composed almost wholly of these, rarely composed of thin, irregular angular plates. Sarcode colorless. Pseudopods digitate. In moss and scum from gully in golf links, Lawrence. Plate XXXVI; fig. 69.

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THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 10—January, 1913.

(Whole Series, Vol. XVII, No. 10.)

CONTENTS:

A Comparison of Naturally and Artificially Aroused Impulses under the Influences of Nerve Block.....*Charles M. Gruber.*

PUBLISHED BY THE UNIVERSITY,
LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 10] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No. 10

A COMPARISON OF NATURALLY AND ARTIFICIALLY AROUSED IMPULSES UNDER THE INFLUENCE OF NERVE BLOCKS.

BY CHARLES M. GRUBER.

From the physiological laboratory of the University of Kansas.

THE research, of which a preliminary report is here presented, was begun by Doctor Hyde several years ago, at the suggestion of Dr. C. S. Sherrington, in his laboratory in the University of Liverpool, and was completed by one of Doctor Hyde's research students.

It was conclusively proved by me that impulses aroused by artificial stimuli in a frog's sciatic nerve can be suppressed by the tripolar electrical block, liquid air, and other blocking agencies, and that the tripolar block could be repeatedly introduced in the circuit without injury to the nerve's function. Also that afferent impulses could be inhibited with weaker currents than could the efferent impulses. It was of interest, therefore, to ascertain whether natural impulses originated by higher centers in the brain were physically of the same nature as those electrically produced, and could be suppressed by the same kind and strength of block as could those artifically aroused by the electrical current.

The experiments were conducted on the phrenic nerves in rabbits. By means of a modification of Head's method, records of the movements of the diaphragm and the thoracic wall were obtained. The artificial stimuli were the threshold strengths of the faradic currents applied to the phrenic nerve and its branches; the movements of the diaphragm in response to

Received for publication November 12, 1912.

these stimuli and the contractions of the diaphragm resulting from the stimuli emanating from higher centers in the brain were recorded and compared. If the phrenic nerve or any of its branches are stimulated with the weakest possible current, it is responded to, throughout the period, by a contraction of the diaphragm or an inspiration phase of the side stimulated. The efficiency of the tripolar electrical block was compared with others, especially those produced by cocaine and freezing mixtures, and liquid air.

It was found that the contractions of the diaphragm were suppressed the instant and throughout the period that the tripolar block was closed. The strength of the block current was not more than $(2.8)^2$ volts, and this strength would suppress the cranial or naturally aroused impulses, as well as those originated by the very weakest faradic current. Only in one experiment did a block of $(1.4)^2$ volts inhibit the cranial but not the artificially aroused impulses.

But cocaine, a freezing mixture of sodium chloride and ice, or liquid air, proved efficient in suppressing the impulses originated by the higher centers in the phrenic as well as those produced by electrical stimuli.

The presence of afferent fibers in the phrenic was well demonstrated by a simple experiment. When the motor impulses on one side were blocked, and the phrenic stimulated centrally to the block, a change in rate and amplitude of the diaphragmatic movements of the opposite followed, as was indicated by the contraction curves secured from the diaphragm slips on that side. The same result was obtained when the phrenic of one side was cut and the central cut end stimulated. Stimulating or irritating the peritoneum in different regions of the abdomen with intact phrenic is followed by a change in rate and force of the respiratory movements. This is, however, probably due to afferent stimuli from nerves other than the phrenic to the respiratory center.

THE

KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 11—January, 1913.

(Whole Series, Vol. XVII, No. 11.)

CONTENTS:

Blocking Nerve Impulses Aroused in Cut and Intact Nerves and in Strychninized and Normal Frogs.... *Charles M. Gruber.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 11] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No 11

BLOCKING NERVE IMPULSES AROUSED IN CUT AND INTACT NERVES, AND IN STRYCHNINIZED AND NORMAL FROGS.

BY CHARLES M. GRUBER.

From the physiological laboratory of the University of Kansas, under the direction of I. H. HYDE, PH. D.

THE object of the investigation, of which a preliminary sketch follows, was to study the practical value of the tripolar block upon impulses aroused by different kinds of stimuli, and to compare its efficiency and advantages with the nerve blocks produced by other methods upon cut and intact nerves in the frog.

From many investigations it has been demonstrated that nerve conduction may be inhibited by the same forces, but of different degree, as those that arouse nerve impulses, and effect stimulation, namely, mechanical (compression), chemical (cocaine and magnesium sulphate), thermal (freezing), and electrical (either faradic or galvanic bipolar or tripolar currents).

The tripolar galvanic current is usually used for stimulation, though it has in a few instances been employed to block vagus impulses. In this investigation the tripolar current was brought to the nerve by interpolating between the nonpolarizable electrodes and the nerve, zephyr fibers, moistened in normal salt solutions.

It was proved from the first that the bipolar method was less efficient than the tripolar, and it was therefore abandoned.

The stimulating current consisted of the minimal faradic strength that caused a maximal contraction of the gastroc-

Received for publication November 12, 1912.

nemious muscle. Such contractions instantly ceased when the block circuit was closed, and reappeared immediately when it was broken.

The strength of the current necessary to block the impulses varied from eight to twenty-five volts, depending upon the kind of stimuli and also upon the condition of the frog. The strongest tetanic contractions produced by stimulating the sciatic nerve with the faradic current, in frogs in good condition, required, as a rule, about twenty volts in the block circuit.

Strychnine tetanus was inhibited with weaker currents, as were also efferent impulses reflexly produced by stimulating either the foot or nerve of the opposite leg.

Any of the legs can be severed without causing reflex contractions in the leg to which the block current was sent, or by placing the tripolar block on its innervatory nerve above the cut.

The tripolar block can be applied repeatedly both to cut and intact nerves, with but slight, if any, after effects upon the functions of the nerve.

Afferent impulses were inhibited by currents of only one-tenth of the strength necessary for efferent fibers.

In every case the efficiency of the block was proved also by the absence of action currents beyond the block.

The advantages of the tripolar block, as compared with the others that were investigated, are readily seen when we consider that it requires about one hour before magnesium sulphate or cocaine produce an efficient block, and they have an injurious after effect upon the function of the nerve. However, they act alike on both cut and intact nerves; and that the freezing mixtures, especially liquid air, require several minutes before they produce blocking, and many more before their effect passes off, although the freezing may be applied repeatedly without injurious after effect upon intact nerves.

But it is interesting to note that freezing the nerves, either before or after cutting them, destroys their function at the blocked area. This fact is one of the conspicuous differences between the different blocks experimented with.

The tripolar galvanic block current, therefore, proved superior to the others in several respects. It may be employed indefinitely on both cut and intact nerves, acts instantaneously,

and its effect is just as quickly removed without injury to the function of the nerves; and what is also of importance is, that we have in it an effective, ready power, by means of which we can exclude the conductivity of the afferent fibers in a mixed nerve trunk.

It is possible that it may prove a practical and valuable aid in surgery and for experimental purposes.

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 12 — January, 1913.

(Whole Series, Vol. XVII, No. 12.)

CONTENTS:

THE INFLUENCE OF ADRENALIN, MODIFIED BY SALT SOLUTIONS, ON BLOOD-PRESSURE IN THE FROG.....*Ivan Roy Burket.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 12] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No.

THE INFLUENCE OF ADRENALIN, MODIFIED BY SALT SOLUTIONS, ON BLOOD-PRESSURE IN THE FROG.

BY IVAN ROY BURKET.

(From the laboratory of the Physiology Department, University of Kansas.)

THE technique for obtaining blood-pressure in the frog is much simpler than that required in the determination of the blood-pressure in mammals. It was, therefore, undertaken to ascertain the relative merits of the two methods in the standardization of adrenalin. In this connection, also, the attempt was made to determine whether or not the action of adrenalin in the body is modified by the presence of various organic and inorganic salts, which may either be present normally in the blood or be otherwise used therapeutically.

Very little work has been done on the blood-pressure in the frog except to determine the effect of external influences. Schulz[1], in his paper on the effect of temperature, vagus stimulation, etc., made some assertions which I have been unable to corroborate and which I will discuss later. His method of operating, however, is the one I found to be most convenient and practical and is used throughout my experiments.

Neither is literature very extensive on the influence of solutions of various salts on mammalian blood-pressure. A few experimenters, however, have published articles along this line of research, and these will be referred to in the discussions of the several salts.

I have been able to find only one paper[2] on the influence

Received for publication December 8, 1912.

1. Schulz: Archiv. für die Gesammte Physiologie, 1906, CXV, 386-447.

2. Frankl: Archiv für die Gesammte Physiologie, 1909, CXXX 346-352.

of salts on the activity of adrenalin; this one deals with the inhibition of the pupil reaction.

EXPERIMENTS UPON THE FROG.

Method.

In preparing the frog for the experiment the animal was first anesthetized with ether, then fastened onto the holder with ordinary binding tape, after which an occasional drop of ether was sufficient to prevent struggling. The lower part of the abdominal aorta and the two iliac arteries were exposed by removing the oscoccygis (see Schulz, *loc. cit.*), after which the canula (three-way for the purpose of cleaning) was inserted into the right iliac when this vessel was large enough, otherwise the abdominal aorta itself was employed. The canula was connected to a Hürthle manometer, using as a transmission fluid the solution suggested by Schulz—10 per cent dextrose and 1 per cent ammonium oxalate. The injections were made into the crural vein of the left leg through a glass canula, which was tied into this vessel and supported by a clamp. It is necessary to support both canulæ in a favorable position, because of the great tenderness of the frog's blood vessels. Throughout the experiment the animal was kept covered with a moist cloth to prevent drying of the skin and other undesirable sensory stimuli, which, according to Schulz, cause a marked change in the blood-pressure.

The respiratory curve was obtained by the use of two tambours—one placed under the animal's head and in contact with the skin over the submental muscles; the other placed on a stand and operating a writing lever. The results obtained were, however, very unsatisfactory, because of the irregular respiration of the frog when the animal is disturbed.

Frogs of about 100 gm. weight were chosen, so that a constant dose of the solutions (0.25 c.c.) could be used.

I had no occasion to test the reflex stimulation of the vagus through the sensory nerves other than the optic, on the effect of which Schulz (*loc. cit.*) makes the following remark: "Es war zur Hervorrufung von Vaguskurven gar nicht einmal nötig, durch auffallende Bewegung in der Nähe der Augen einen Reiz auszuüben, sondern es genügte das erscheinen eines fremden Gegenstandes innerhalb des Gesichtsfeldes schon auf weite Entfernung hin." In view of this statement, and because it is difficult to avoid slight changes in light and other

elements of the frog's visual field, I took pains to make the following test on several occasions, both before any injection had been made and also between injections. The moist cloth with which the animal was kept covered, as before stated, was lifted, and the eye exposed to the light of a 16-candle-power electric bulb for one minute at a time. With even this strong stimulation there was absolutely no change in the pressure curve in any experiment. One would expect, however, to observe a slight change in heart action due to reflex vagus stimulation, and in a few instances there was a slowing of one to four beats per minute; but this effect was by no means a constant result of the light stimulation.

It was found that, because of the peculiar action of adrenalin in the frog, definite results could not be obtained from injection of salts and adrenalin simultaneously; therefore this discussion will deal with the separate action of these two classes of substances.

A comparison of tables I and III, pertaining to the effect of salts, will show that when the doses of the different salts were properly regulated, the salts under consideration had practically the same effect on the frog as on the cat.[3] With two exceptions—KCl and $BaCl_2$—equimolecular solutions were used on both animals to produce these similar effects, the dose being varied as to quantity only (0.25 c.c. for the frog and 1 c.c. for the cat); of the two salts mentioned, a stronger solution of KCl was required to bring about a depression in the frog's blood-pressure, while a much weaker one of $BaCl_2$ was necessary to produce the heart stimulation and rise of pressure. M/32 barium chlorid was toxic to the frog, and would quite often entirely stop the heart for several seconds; M/128 $BaCl_2$ had the usual tonic action.

It was in working out the initial object of this research (that of determining the value of the frog's blood-pressure as a means of standardizing adrenalin) that obstacles were encountered which would seem to rule out the frog as a reliable animal for this kind of work. The initial dose of adrenalin would always cause a considerable rise in pressure—sometimes as high as 12-15 mm.—but the effect of succeeding injections declined rapidly till the fourth or fifth dose, which would cause a change of probably 1-3 mm., this effect remain-

3. Burket, I. R. Amer. Journal of Physiology, 1912, XXX, IV, p. 382.

ing nearly constant for several doses, the size of the dose thereafter making but little difference. (The solutions used were: 1:100,000, 1:60,000, 1:30,000, and 1:10,000.) The rise in pressure lasted considerably longer than in the cat, and there was no after depression as in the cat, due, as was proved later, to impurities in the Parke-Davis adrenalin. But the peculiar decrease in susceptibility to the drug rendered it impossible to determine with any degree of accuracy the strength of adrenalin used, or in the other part of the research, to ascertain the effect of the salts upon adrenalin pressure.

This work was done at the suggestion of Doctor Hyde, to whom my best thanks are due for her valuable advice and help.

Table No. I. Cat – Salt Effects.

Amount Injected: 1 cc of each solution.

Solution.	Strength.	Amount of salt injected. *mgm.*	Effect on blood-pressure.	Effect on heart action.
Adrenalin (P. D. & Co.)	1:10,000	0 01	Increase (ave., 64 mm. Hg.); then a fall, due to impurities in P. D. Co., adrenalin	Rate decreased; force greatly increased.
Ringer			Neutral	None.
NaCl	M/8	7.3	Neutral	None.
NaCl	M/2	29.0	Slight increase (ave., 2 mm.)	Undetermined.
KCl	M/32	2.2	Slight decrease (ave., 2 mm.)	Undetermined.
KCl	M/16	4.5	Slight decrease (ave., 4 mm.)	Rate slightly decreased.
$CaCl_2$	M/32	3.4	Neutral	None.
$CaCl_2$	M/16	6.8	Slight decrease (ave., 4 mm.)	Not determinable.
$MgSO_4$	M/8	15.0	Decrease (ave., 9 mm.)	None or slight increase in rate.
$BaCl_2$	M/128	0.4	Marked increase (ave., 15 mm.)	Rate slight decrease; force increase.
$BaCl_2$	M/32	1 6	Marked increase (ave., 23 mm.)	Rate markedly decreased; force increased.
Uric acid	0.005%	0 01	Increase (ave., 9 mm.)	None or slight increase in weight.
NaH_2PO_4	M/16	6 0	Slight increase (ave., 5 mm.)	Undeterminable.
NaH_2PO_4	M 16	8.8	Neutral	Undeterminable.
Na_3PO_4	M/16	10.0	Variable	Undeterminable.
HCl*	0.35%		Very slight increase	Not noted.
NaCH†	0.3%		Neutral or very slight increase	Not noted.

* Same acidity as M/16 NaH_2PO_4.

† Same alkalinity as M/16 Na_3PO_4.

Table No. II. Cat—Adrenalin and Salts.

Amount Injected: 1 cc. each of adrenalin and salt solution.

Adrenalin* and following.	Strength.	Effect on adrenalin pressure.	Effect on heart action.
Adrenalin alone	1:100,000	Ave., 35 mm. Hg.	Rate decreased; force increased.
Adrenalin alone	1:10,000	Ave., 64 mm. Hg.	Same as above, but more marked.
Ringer		No change.	Same as adrenalin alone.
NaCl	M/8	Variable	Same as adrenalin alone.
NaCl	M/2	Decrease (ave., 5 mm.)	Not determined.
KCl	M/32	No change.	Not determined.
KCl	M/16	Increase (ave., 4 mm.)	Rate and force both decreased.
$CaCl_2$	M/16	Decrease (ave , 9 mm.)	Same as adrenalin alone.
$MgSO_4$	M 8	Decrease (ave., 13 mm.)	Force increased—less than by adrenalin alone.
$BaCl_2$	M/128	Increase (ave., 9 mm.)	Augments adrenalin effect.
$BaCl_2$	M/32	Increase (ave., 25 mm.)	More marked than M/128.
Uric acid	0 005%	Decrease (ave , 12 mm.)	Rate and force decreased.
NaH_2PO_4	M/16	No change	Undetermined.
Na_2HPO_4	M/16	No change	Undetermined.
Na_3PO_4	M/16	Decrease (ave., 17 mm.)†	Undetermined.
HCl	0 004%	No change	Same as adrenalin alone.
NaOH	0 004% HCl	No change	Same as adrenalin alone.

* Strength of adrenalin used was in nearly all cases 1:10,000.

† This marked effect was probably due to oxidation of the adrenalin, which takes place rapidly in alkaline solution.

Table No. III.—Salt Effects on the Frog.

Amount injected: 0.25 cc. of each solution.

Solution.	Strength.	Amount of salt injected.	Effect on blood-pressure.	Effect on heart action.
		mgm.		
Ringer			Neutral	None.
NaCl	M/8	1.8	Neutral	None.
NaCl	M/2	7.3	Neutral	None or slight decrease in rate.
NaCl	M/1	14.6	Decrease (ave., 9 mm.)	None or slight decrease in rate.
KCl	M/16	1.1	Neutral	None
KCl	M/8	2.2	Marked decrease	Nearly stopped for few seconds.
$CaCl_2$	M/16	1.7	Decrease (ave., −4 mm.)	Rate slightly decreased.
$MgSO_4$	M/8	3.75	Decrease (−3 mm.)	None apparent.
$BaCl_2$	M/128	0.4	Increase (3 mm.)	Rate decreased; force increased.
$BaCl_2$	M/32	1.6	Decrease (−8 mm.)	Same as M/128, but more marked.*
Uric acid	0.005%	0.01	Increase (1 mm.)	None apparent.
Na_2HPO_4	M/16	2.20	Variable	Variable.
NaH_2PO_4	M/16	1.75	Decrease (−4 mm.)	Variable.
Na_3PO_4	M/16	2.5	Decrease (−4 mm.)	Not determined.
HCl	0.004%		Slight increase (ave. 1 mm.)	None.
NaOH	0.007%		Neutral	None.
Adr. 1st.	1:30,000		Increase (12 mm.)	
Adr. 2d.	1:30,000		Increase (7 mm.)	
Adr. 3d.	1:30,000		Increase (4 mm.)	
Adr. 4th.	1:30,000		Increase (3 mm.)	
Adr. 5th.	1:10,000		Increase (3 mm.)	

* This was after the initial toxic effect had passed.

CONCLUSION.

The frog's blood-pressure is not a reliable means of standardizing adrenalin, because the frog shows a decrease in susceptibility to the drug after the first or second injection.

With the exceptions of KCl, which required a stronger solution to produce depression, and $BaCl_2$ which required a weaker solution to increase pressure and heart action, the salts experimented with had practically the same effect upon the frog as they had upon the cat.

THE

KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 13 — June, 1913.

(Whole Series, Vol. XVII, No. 13.)

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PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter.

4–5259

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THE INFLUENCE OF ALCOHOL UPON THE REFLEX ACTION OF SOME CUTANEOUS SENSE ORGANS IN THE FROG.

BY IDA H. HYDE, RUTH SPRAY, AND IRENE HOWAT.

(From the Physiological Laboratory of the University of Kansas.)

With one text figure

THE following questions form the basis of the present research: How soon after administering a minimal and and also stronger dose of alcohol does a change in reflex time appear? How long does the change last? When do the reflex actions cease? For how long a period are they absent? When does the reflex time become normal again? The experiments were conducted on both spring and normal frogs of the species *Rana esculenta* and *pipiens* that were secured from a pond near by. They weighed from 25 to 60 grams and possessed well-marked pigment spots. During the whole period of experimentation the frogs were kept in moist moss in high glass jars, the sides of which were covered with dark paper.

Careful testing of most of the pigment spots revealed the fact that they may be regarded as peripheral sensory organs. Those on the head are innervated by branches from the cranial nerves, and those on the trunk and legs by spinal nerves. Moreover, some of these spots display a fairly constant reflex time to a definite stimulus when the frog is in a good condition, and others are peculiarly irritable. The same spot also varies more or less in its reaction time at different times of the day; so that each one may be said to have its own period of

Received for publication April 23, 1913.

reflex time. Those spots were finally selected for study that proved after prolonged testing most reliable and constant in their reactions to a definite stimulus. The position of the chosen spots are illustrated in text, figure 1.

It was important to find a stimulus that was of a constant definite strength and that would neither injure nor fatigue the peripheral nerve ending. Pure neutral filter paper three millimeters square and moistened with 8 per cent pure acetic

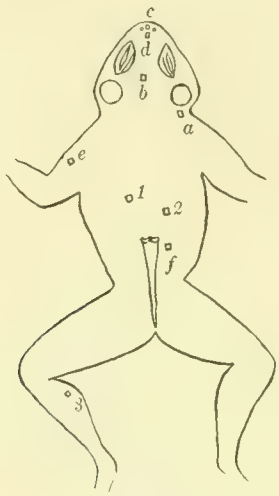

FIGURE 1.

acid was chosen. The acid paper was placed upon the desired spots with great care by means of a long, delicate forceps, so as to exclude sight and pressure stimuli. That these errors were avoided was proved by control experiments, during which the frog's eyes were carefully covered by a specially devised hood.

By the reflex time is meant the interval between the moment the paper touches the skin and the moment the frog made an attempt to remove it. It was found that if the acid was not rubbed off within one minute it never would be. As soon as the attempt was made, or if it was not made in one minute, the spot was washed off with fresh water. Thus fatigue and injury to the nerve ending in the spot was prevented, as was proved by testing the corresponding spot on the opposite side of the body that had not been stimulated before. By this

means, and also from control experiments on unalcoholized frogs with 8 per cent acetic acid, it was proved that under the conditions of experimentation fatigue of the peripheral nerve ending due to the stimulus played a subordinate part in the results.

Fixed doses of pure alcohol, varying from 10 to 95 per cent, were experimented with. The dose determined upon for the different investigations consisted of a definite number of cubic centimeters of the desired per cent of alcohol for 10 grams frog.

The alcohol was given per stomach, either in gelatine capsules or through a pipette, or by subcutaneous injections into the lateral lymph sack. The last method proved most satisfactory, because this manner of administration annoyed the frog less, and it was essential to avoid unnecessary excitement or stimulation, in order to keep the conditions uniform throughout. The alcohol has its depressing effect about five minutes earlier when given per stomach, being absorbed more readily than when given subcutaneously.

The method of experimentation required that the reflex time of each one of the sensory pigment spots of a series be first secured from an unalcoholized frog. The frog that was to be experimented on, moreover, must have shown from repeated tests that the responses were constant for long periods of time. The reflex time for each spot having been secured, the desired dose of the alcohol was given, and at once every spot of the series was tested. This procedure was repeated every ten minutes for about two hours of experimentation, and again six or twenty-four hours later, allowing an interval of ten minutes rest between each test. The turn-over, compensatory and swimming reflexes were also tested. Table I illustrates the method of recording the observations. While one of us recorded the reaction time, the other applied the acid to the spots and swabbed it off immediately after the frog reacted. If it did not react, the acid was washed off after one minute.

It was of interest to find the weakest dose of alcohol that produced a change in the reflex time of the cutaneous sensory spots, and whether such a dose had any apparent effect on other reflex actions, such as turn-over, compensatory and swimming, or upon the general behavior of the frog; also to

learn whether increasing doses of alcohol slowed or quickened the reaction time; also the duration of the effect and whether it varied with different intervals of time. The doses varied from .05 cc. of 15 per cent per 10-gram frog up to 0.6 cc. of 50 per cent, and even fatal doses of 1 cc. of 95 per cent alcohol per 10-gram frog were given.

The reflex time so secured was compared with the reaction time secured before alcohol was given, and in a few cases with those secured after injecting Ringer solution in amounts usually given in alcohol. Tests were also made to see if the mechanical stimulus of the needle would be a disturbing factor. From these observations the influence of alcohol upon the reflex centers of some of the different cutaneous sensory spots was deduced.

It was demonstrated that each spot had its own reflex time and its own degree of irritability, and that some were more resistant to the influence of alcohol than others. Spot *c*, figure 1, for instance, was especially resistant; the acid paper rubbed off within one second, and weak doses of alcohol that affected the reflex time of all the other spots produced no effect upon this spot. Not until a dose of 0.3 cc. of 30 per cent alcohol per 10-gram frog was given did the spot fail to respond, and then only for a short time. Larger doses, 0.6 cc. of 50 per cent alcohol per 10-gram frog, that caused a loss of the reflexes of the other spots and a depression of the turnover and other higher reflexes, would, however, cause a loss of response from this spot as well.

It was shown that alcohol in quantities so small that it apparently produced no effect whatever upon the frog has a decided influence upon the reflex centers of some of the cutaneous areas; at least in so far as to lower their irritability and to prolong their reaction time. When once affected by alcohol, they do not become normal in their response again for some time.

From repeated tests in a large series of experiments it was seen that doses less than 0.05 cc. of 15 per cent alcohol per 10-gram frog had no more effect upon the reflex time than had an equal amount of Ringer solution. But beginning with 0.1 cc. of 15 per cent alcohol, all the other doses that were given had a depressing effect upon the reflex actions; that is, they caused the interval between stimulations and response to be prolonged

or the reflex action to be lost or become irregular in its response within ten minutes. This change lasted from one to a half-hour after injecting the alcohol. Often 0.1 cc. of 15 per cent alcohol and 0.1 cc. of 30 per cent alcohol produced no difference in the same frogs. The effect of the dose, it seems, depends largely upon the condition of the frog and the susceptibility of the special sensory spot tested. With doses beginning with 0.3 cc. of 15 per cent alcohol to about 0.3 cc. of 30 per cent alcohol, the turn-over, compensatory and equilibrium reflexes are not depressed, but the frog seems very irritable and restless, often failing to remove the stimulus in its attempts to jump out of the jar. With doses of 0.3 cc. of 30 per cent alcohol the frog immediately gets sluggish and greater depression of the reflex time is produced, not only in the skin reactions, but also in the higher reflexes; even spot *c* sometimes fails to react. Doses of 0.5 cc. of 30 per cent alcohol cause a loss of all skin reflexes and of muscle tone. Frequently, following such a dose the frog lies flat and is sluggish, has labored respiration, and the turn-over, equilibrium and other higher reflexes are lost for five minutes or more. Later, in some cases, the frog moves about restlessly and makes only weak attempts to remove the stimulus, or it may seem apparently lifeless and all skin reflexes abolished for about one hour. With larger doses of 0.6 cc. of 50 per cent alcohol, the turn-over and higher reflexes are lost within ten minutes and for one to two hours, but in twenty-four hours they are again normal, not, however, the skin reflexes. Doses of 0.8 cc. of 95 per cent alcohol proved toxic.

It was observed that when the spots fail to respond after a given dose they do so immediately, and the depression increases and is of longer duration with increase in dose, until with one of 0.6 cc. of 50 per cent alcohol, when the skin and higher reflexes are all lost for from one to two hours and the skin reactions are not normal again even after twenty-four hours. Sometimes, following such a dose, respiration ceases and tetanic convulsions appear that last for about one hour.

In connection with these experiments it was of interest to determine what some of the doses of alcohol per gram animal would amount to for man. Sherry, orange and port wine contain from 15 per cent to 20 per cent alcohol, claret and white wine from 10 per cent to 14 per cent, beer from 4 per cent to

10 per cent, with a large amount of other ingredients, while whisky and brandy contain from 40 per cent to 50 per cent.

A dose of 0.1 cc. of 15 per cent alcohol per 10-gram frog, which would produce no preceptible change in the behavior of the frog, but would cause an immediate depression, and in some instances a loss of reflex action that lasted for an hour or more, and not giving place to normal after six hours, would be equivalent to 1 pint of sherry or orange or port wine or 1.13 pints of claret or 2 pints of strong beer for a man of average weight, or about 64.7 kilos.

In general, we may say that alcohol in small quantities, even when no perceptible change in the behavior of the frog is noticed, lowers their reaction time. When the reflex centers have once been affected, even slightly, they do not become normal again for six to twenty-four hours. Therefore, even small quantities of alcohol exert a depressing chemical action upon certain parts of the nervous system, which effects may indirectly affect the muscle tone and vasa motor and cutaneous thermal reflex actions more or less seriously. The larger the dose the greater the effect and the quicker it comes on after giving the alcohol. The effect of the alcohol depends somewhat upon the individual. Some frogs become excited, others sluggish; but there was always a depression of the nervous reflex centers. Certain cutaneous nerve organs in the frog are more irritable, and also more susceptible to the influence of alcohol, than others. The skin reflexes are affected by smaller quantities of alcohol than the turn-over, swimming, compensatory and equilibrium reflexes are. It also takes them longer to become normal again. Alcohol taken in sufficient quantities, 0.6 cc. of 50 per cent, for instance, causes loss of all reflexes, a loss of muscle tone, unconsciousness and convulsions similar to those caused by strychnine poisoning.

SUMMARY.

1. Certain sensory spots in the frog's skin differ not only in irritability and reflex action, but in susceptibility to the influence of alcohol, and they vary more or less in their reflex time at different intervals of the day.

2. The sensory spots or cutaneous sensory organs that were experimented on are affected by much smaller quantities of alcohol than are the turn-over, swimming, compensatory or equilibrium reflexes. When the dose is sufficient to produce

any effect at all, it is always a depressed or slowed response to the stimulus, never causing a stimulating effect or shorter reflex time.

3. Doses less than 0.05 cc. of 15 per cent alcohol per 10-gram frog produce no more change in the reflex time of some of the cutaneous sensory organs than does the same dose of Ringer solution. But a dose of 0.3 cc. of 30 per cent, though seeming to stimulate by causing the frog to become irritable and active, nevertheless produces a depression, or causes irregular responses, or an entire loss of the reflex action. These changes are observed in all of the spots experimented on excepting *c*, which is not influenced by weak doses.

Moreover, these changes appear within ten minutes after injecting the alcohol, and persist for from one to one and a half hours. Doses above 0.3 cc. of 30 per cent to 0.6 cc. of 50 per cent produce within five minutes great changes in the frog's behavior and appearance. Often the frog assumes a flat, sluggish and dazed attitude. Both the cutaneous and higher reflexes are lost. The muscle tone is much decreased, respiratory activity is not perceptible, and tetanic convulsions may set in with the stronger doses. This condition may continue from one-half to two hours, but the reflex time of the cutaneous sensory organs do not become normal again even after twenty-four hours.

Alcohol in quantities too small to produce any apparent effect upon the frog has a decided influence upon the reflex centers of some of the cutaneous sense organs, at least in so far as to lower their irritability and reaction time. When these centers are once affected by alcohol they do not become normal again in their reflex responses for from six to twenty-four hours.

When the sensory spots fail to respond to a dose, they do so immediately, and the depression increases, and is of longer duration with increase in dose, until with 0.6 cc. of 50 per cent alcohol, when all of the sensory spots, and the higher reflexes as well, are either much prolonged or lost entirely for several hours, the length of their absence depending upon the condition of the frog experimented on. One cc. of 95 per cent alcohol is toxic.

It was observed that even small quantities exert a depressing chemical action upon certain parts of the nervous system,

TABLE I. A Type Table of Observations.

1. TIME.	2. Time of application.	3 Reaction time.	4. Reflex time.	5. Spot (compare fig. 1).	Frog, 41.6 gms. March 28.
10:00 A. M...	20	23	3	1	Stimulus 0.8% acetic. Reflex time in seconds, from normal frog.
	3	4	1	2	
	6	11	5	3	
	43	44	1	*a*	
	2	7	5	*b*	
	33	34	1	*c*	
	1	13	12	*d*	
	41	42	1	*e*	Turn-over, swimming, compensatory and
4:30 P. M...	55	59	4	1	equilibrium reflexes normal.
	25	26	1	2	
	11	14	4	3	
	29	33	3	*a*	
	51	53	2	*b*	
	12	13	1	*c*	
	19	20	11	*d*	
	45	46	1	*e*	
4:43 P. M...	.....		..	..	Injected a dose of 0.1 cc. per 10 grms. of
4:48 P. M. .	43	56	13	1	30 per cent alcohol.
	26	31	3	2	
	43	25	40	3	
	32	....	-	*a*	(–) Means waited 1 minute, it failed to
	10	19	9	*b*	react.
	45	46	1	*c*	
	12		-	*d*	
4:56 P. M...	27	50	23	*e*	Turn-over, swimming, compensatory and
5:00 P. M...	60	5	5	1	equilibrium reflexes normal.
	38	42	4	2	
	56			3	
	35	.		*a*	
	36	. . .	1	*b*	
	20	21		*c*	
	50	.....		*d*	
5:10 P. M...	26	40	14	*e*	
5:20 P. M...	11	17	6	1	
	2	15	13	2	
	5	....	–	3	
	26	.	.	*a*	
	46	.. .		*b*	
	59	60	1	*c*	
	17	.. .		*d*	
	35	.....		*e*	
5:45 P. M...	29	41	12	1	
	23	35	12	2	
	5	12	8	3	
	45		–	*a*	
	46	..	–	*b*	
	55	8	13	*c*	
	2	.....	-	*d*	
	2	17	15	*e*	
6:03 P. M...	5	12	8	1	
	42	48	6	2	
	2	18	16	3	
	8	.	–	*a*	
	15	.....	–	*b*	
	59	5	6	*c*	
	40	.		*d*	
	48	52	4	*e*	
10:25 A. M...	60	11	11	1	March 29.
	44	47	2	2	
	8	13	5	3	
	57	5	8	*a*	
	28	35	7	*b*	
	50	52	2	*c*	
	30	31	1	*d*	
	47	48	4	*e*	
	34	43	9	1	
	4	11	7	2	
	1	15	14	3	
	13	.....	–	*a*	
	52	54	4	*b*	
	16	17	1	*c*	
	42	.....	–	*d*	
	29	42	13	*e*	

and affect the muscle tone, vaso motor and cutaneous thermal reflex actions to a degree depending upon the susceptibility and condition of the nervous centers.

Doses of 0.1 cc. of 15 per cent alcohol per 10-gram frog would be equivalent to one pint of sherry or orange or port wine, or 1.13 pints of claret or two pints of strong beer for a man of average weight. This dose caused in the frog a depression of the reflex time of the cutaneous sensory organs.

Explanation of Table I.

A copy of a table of observations chosen at random, 0.8 per cent acetic acid used as a stimulus.

Column 1 records at intervals the time of day, in hours and minutes, when the experiment was conducted, first on the unalcoholized and later on the alcoholized frog.

Column 2 records the reading of the second hand of the watch when the acid paper was applied.

Column 3 records the reading of the second hand when the frog attempted to rub off the paper.

Column 4 gives the reflex time; that is, the seconds elapsing between the application of the acid and the attempt of the frog to rub it off.

Column 5 indicates the area on which the acid paper was put. See figure 1. (—) means that the frog failed to react within one minute.

From such data we would conclude:

1. That this particular frog had been at the beginning of the experiment quite normal, in that it responded well to the stimulation and quite constantly, the reaction time being about the same for the same spots, even at quite different times of the day (10 A. M. and 4:30 P. M.).

2. That the reflex centers of most of the spots had been immediately affected by injecting 30 per cent alcohol in the amount of 0.1 cc. per 10 grams of its weight, *c* being the only spot not quickly affected.

3. That spots *a* and *b* were most quickly and longest affected, not reacting once within about an hour and a half after the injection, at which time the experimenting was stopped for that day. That *c* was least affected, being one of the spots which never failed utterly to react.

4. That the frog was not quite normal the following day, reacting somewhat more irregularly than normally.

5. That this dose is not sufficient to produce any apparent effect on the turn-over, swimming, compensatory and equilibrium reflexes.

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THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 14.—January, 1913.

(Whole Series, Vol. XVII, No. 14.)

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COLLINEATIONS IN SPACE OF FOUR DIMENSIONS.....*R. S. Pond.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post office in Lawrence as second-class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 14] JANUARY, 1913. [WHOLE SERIES VOL. XVII, No. 14.

COLLINEATIONS IN SPACE OF FOUR DIMENSIONS.*

BY R. S. POND.

THE object of this paper is to outline the theory of collineations in space R_4 of four dimensions; in particular, to derive what Professor Newson has called the "normal forms," and to deduce the twenty-seven types of collineations which exist in the R_4.

The subject is introduced by a brief discussion of the ordinary analytic form of a collineation in point coordinates with regard to its geometrical interpretation, and of the dependent transformations which line, plane and space coordinates undergo. The geometrical terminology followed is that of M. Jouffert, in his *Traite de Geometrie a Quatre Dimensions.*

*Received for publication September 23, 1912.

Part I.

General analytic form of collineation in the R_4.

§ 1. Consider the substitution T:

$$X' = \frac{a_{11}x + a_{12}y + a_{13}z + a_{14}u + a_{15}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{X}{V}$$

$$Y' = \frac{a_{21}x + a_{22}y + a_{23}z + a_{24}u + a_{25}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{Y}{V}$$

$$Z' = \frac{a_{31}x + a_{32}y + a_{33}z + a_{34}u + a_{35}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{Z}{V}$$

$$U' = \frac{a_{41}x + a_{42}y + a_{43}z + a_{44}u + a_{45}}{a_{51}x + a_{52}y + a_{53}z + a_{54}u + a_{55}} = \frac{U}{V}$$

This substitution changes the set of numbers x, y, z, u, into the set x', y', z', u'. If we take x, y, z, u, to be the Cartesian coordinates of a point in the R_4 referred to four mutually perpendicular spaces as spaces of reference, then the substitution transforms the point whose coordinates are x, y, z, u, into the point x', y', z', u'. If we solve the four equations of T for x, y, z, u, we get a transformation of the same form as T, viz., T^{-1}:

$$X = \frac{A_{11}x' + A_{12}y' + A_{13}z' + A_{14}u' + A_{15}}{A_{51}x' + A_{52}y' + A_{53}z' + A_{54}u' + A_{55}} = \frac{X'}{V'}$$

with three similar expressions for y, z, and u, where A_{ij} is the co-factor of a_{ij} in the matrix

$$(a) = \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{matrix}.$$

Evidently T and T^{-1} set up a one-to-one correspondence between the points of the R_4. T^{-1} always exists if $\triangle \neq 0$, $\triangle$ being the determinant of the matrix (a). By T the point $X = Y = Z = U = 0$ is transformed into the origin $x' = y' = z' = u' = 0$. The origin goes over into the point

$$\left(\frac{a_{15}}{a_{55}}, \frac{a_{25}}{a_{55}}, \frac{a_{35}}{a_{55}}, \frac{a_{45}}{a_{55}}\right).$$

All points of the space $V = 0$ go into the points of the space at infinity.

Since T^{-1} transforms points of the space $V' = 0$ to infinity, T must transform the points of the space at infinity into the points of $V' = 0$. T and T^{-1} may be written in homogeneous form as follows:

(1) T: $\rho x'_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 + a_{i5}x_5$,

(2) T^{-1}: $\rho x_i = A_{1i}x'_1 + A_{2i}x'_2 + A_{3i}x'_3 + A_{4i}x'_4 + A_{5i}x_5 \qquad (i = 1, 2, 3, 4, 5)$,

where A_{ij} has the same meaning as before.

§ 2. T transforms lines into lines; for if x, y, and z, are three collinear points, a relation exists:

$$Z_i = c_1 x_i + c_2 y_i \qquad (i = 1, 2, 3, 4, 5)\ .$$

Transform the three points by T into x', y', and z'.

Then it appears that the same linear relation exists between x_i', y_i', and z_i', namely:

$$Z_i' = c_1 x_i' + c_2 y_i'\ ,$$

which means that x', y', and z' are collinear points also, and therefore lines go over into lines under T, and hence T is a collineation. A similar argument shows T sends planes into planes and spaces into spaces.

§ 3. By means of T we can transform any six points of the R_4 into any other six points; for, if we lay down the conditions that a set of six given points be transformed into any other six points by T we obtain a set of thirty equations in thirty-one unknowns: the twenty-five parameters of (a) and six ρ's.

There is a solution of these 31 equations for which no ρ is o and all other solutions are proportional to this one. Hence six points can be transformed into any other six points in one and only one way, and in doing so we determine the collineation uniquely.

§ 4. What points, if any, are left invariant under T? Any such points must satisfy the conditions

$$\rho x_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 + a_{i5}x_5 \qquad (i = 1, 2, 3, 4, 5),$$

or,

$$(3)\qquad \begin{aligned} (a_{11}-\rho)x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 &= o \\ a_{21}x_1 + (a_{22}-\rho)x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 &= o \\ a_{31}x_1 + a_{32}x_2 + (a_{33}-\rho)x_3 + a_{34}x_4 + a_{35}x_5 &= o \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + (a_{44}-\rho)x_4 + a_{45}x_5 &= o \\ a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + a_{54}x_4 + (a_{55}-\rho)x_5 &= o\,. \end{aligned}$$

The necessary and sufficient condition that this system of equations has other solutions than $x_1 = x_2 = x_3 = x_4 = x_5 = o$ is that their eliminant $= o$.

This is a quintic in ρ and in general has five distinct roots, for each one of which we get a separate solution of (3).

Hence, in general, T leaves invariant a figure consisting of five distinct points and their joins, which are ten lines, ten planes, and five spaces.

The equation in ρ is called the characteristic equation of the collineation, and in a later section we shall investigate it more closely to discover what variations may occur in this invariant configuration.

Part II.

Dependent Transformations.

§ 5. By the same argument that is used for space of lower order we can show that the ten determinants of the second order of the matrix

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{vmatrix}$$

determine uniquely the line through the points x and y and can be used as the homogeneous coordinates of the line.

Likewise the third-order determinants of

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{matrix}$$

may be used as homogeneous coordinates of the plane through the three points x, y, and z, and the five fourth-order determinants of

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{vmatrix}$$

can be used as homogeneous coordinates of the space determined by the four points x, y, z, and u. Denote these line, plane, and space coordinates by $p_{i,j}$, $q_{i,j,k}$, and $s_{i,j,k,l}$, respectively.

When we transform our R_4 by T the p's, q's, and s's undergo certain transformations dependent on the T. The form of these dependent transformations we shall now show.

Suppose x and y go into X and Y, respectively, under T. Then the line whose coordinates are the p's goes into a line whose coordinates are

$$P_{ij} = \begin{vmatrix} X_i & X_j \\ Y_i & Y_j \end{vmatrix}$$

$$= \begin{vmatrix} a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 + a_{i5}x_5 & a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + a_{j4}x_4 + a_{j5}x_5 \\ a_{i1}y_1 + a_{i2}y_2 + a_{i3}y_3 + a_{i4}y_4 + a_{i5}y_5 & a_{j1}y_1 + a_{j2}y_2 + a_{j3}y_3 + a_{j4}y_4 + a_{j5}y_5 \end{vmatrix}$$

which turn out to be

(4)
$$P_{ij} = \sum \begin{vmatrix} a_{ir} & a_{jr} \\ a_{is} & a_{js} \end{vmatrix} p_{rs}$$

$$\begin{matrix} i = 1 \ldots 5 & r = 1 \ldots 5 \\ j = 1 \ldots 5 & s = 1 \ldots 5 \\ i < j & r < s \end{matrix}$$

a linear transformation on the p's whose coefficients are second-order determinants of (a) and whose determinant $= \Delta^4$.

These ten equations are not independent of one another, for, expanding the vanishing fourth-order determinants of the matrix

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{vmatrix},$$

we find that the p's satisfy the following five conditions:

$$\begin{aligned}
&(1)\quad p_{12}\,p_{34} + p_{13}\,p_{42} + p_{14}\,p_{23} = 0\\
&(2)\quad p_{12}\,p_{35} + p_{13}\,p_{52} + p_{15}\,p_{23} = 0\\
&(3)\quad p_{12}\,p_{45} + p_{14}\,p_{52} + p_{15}\,p_{24} = 0\\
&(4)\quad p_{13}\,p_{45} + p_{14}\,p_{53} + p_{15}\,p_{34} = 0\\
&(5)\quad p_{23}\,p_{45} + p_{24}\,p_{53} + p_{25}\,p_{34} = 0\,.
\end{aligned}$$

Now any pair of these five equations is equivalent to any other pair. Multiply (1) by p_{35}, (2) by p_{34}, and subtract. This gives

$$(1)\quad p_{14}\,p_{12}\,p_{35} + p_{14}\,p_{23}\,p_{35} + p_{13}\,p_{25}\,p_{34} + p_{15}\,p_{12}\,p_{34} = 0\,.$$

Then if we multiply (4) by p_{23}, (5) by p_{31}, and subtract, we get identically (1).

Hence the p's are subject to three independent conditions.

§ 6. In the same manner it can be shown that the q's undergo a dependent linear transformation, whose coefficients are the third-order determinants of (a) and whose determinant $= \triangle^6$. The ten equations of this transformation are not independent, but are tied up to the same extent as the equations on the p's.

Similarly we have a transformation on the s's of which the coefficients are the fourth-order determinants of (a) and whose determinant is $= \triangle^4$.

Part III.

Collineations in Normal Form.

§ 7. In this section we shall develop through geometrical considerations our collineation in what Professor Newson has called the normal form. In this form the parameters of the collineation are explicitly

the natural parameters of the geometrical transformation, namely: the coordinates of the invariant points and the characteristic cross-ratios along the invariant lines.

Let a, b, c, d, and e be the five points invariant under our collineation, having coordinates $(a_1 a_2 a_3 a_4)$, $(b_1 b_2 b_3 b_4)$, etc.

Suppose P and Q, having coordinates $(x_1' x_2' x_3' x_4')$ and $(x_1 x_2 x_3 x_4)$, are two corresponding points under our transformation. Let PO, QS, PM, and QR be perpendiculars from P and Q to the spaces $bcde$ and $acde$, respectively.

Now, $acde$ and $bcde$ are the double spaces of a one-dimensional pencil of spaces through the plane cde, and $Pcde$ and $Qcde$ are two corresponding spaces of this pencil.

The cross-ratio of this pencil is

$$\frac{PM}{PO} : \frac{QR}{QS}.$$

Corresponding spaces of this pencil will cut the line ab in corresponding points of the projective range of points on ab, and the cross-ratio of the pencil of spaces must be the same as the cross-ratio along the line. Call this ratio Kab, then:

$$\frac{PM}{PO} : \frac{QR}{QS} = Kab.$$

But $PM:QR$ and $PO:QS$ are the ratios of the volume of the pentahedroides $Pcdea : Qcdea$ and $Pbcde : Qbcde$. Then we have:

$$\frac{Pcdea}{Pbdea} : \frac{Qcdea}{Obdea} = \frac{PM}{PO} : \frac{QR}{QS} = Kab,$$

or

$$\begin{vmatrix} x_1' & x_2' & x_3' & x_4' & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ e_1 & e_2 & e_3 & e_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \end{vmatrix} \qquad \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 \\ c_1 & c_2 & c_3 & c_4 & 1 \\ d_1 & d_2 & d_3 & d_4 & 1 \\ e_1 & e_2 & e_3 & e_4 & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \end{vmatrix}$$

(5) $$\frac{\begin{vmatrix} x_1' x_2' x_3' x_4' & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \\ c_1 \; c_2 \; c_3 \; c_4 & 1 \\ d_1 \; d_2 \; d_3 \; d_4 & 1 \\ e_1 \; e_2 \; e_3 \; e_4 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 \; x_2 \; x_3 \; x_4 & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \\ c_1 \; c_2 \; c_3 \; c_4 & 1 \\ d_1 \; d_2 \; d_3 \; d_4 & 1 \\ e_1 \; e_2 \; e_3 \; e_4 & 1 \end{vmatrix}} = Kab,$$

and we can deduce in the same manner three other equations:

$$\frac{\begin{vmatrix} x_1' x_2' x_3' x_4' & 1 \\ d_1 \; d_2 \; d_3 \; d_4 & 1 \\ e_1 \; e_2 \; e_3 \; e_4 & 1 \\ a_1 \; a_2 \; a_3 \; a_4 & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \end{vmatrix}}{D'} = Kac \qquad \frac{\begin{vmatrix} x_1 \; x_2 \; x_3 \; x_4 & 1 \\ d_1 \; d_2 \; d_3 \; d_4 & 1 \\ e_1 \; e_2 \; e_3 \; e_4 & 1 \\ a_1 \; a_2 \; a_3 \; a_4 & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \end{vmatrix}}{D},$$

(D' and D are the denominators of left and right side of (5), respectively.)

$$\frac{\begin{vmatrix} x_1' x_2' x_3' x_4' & 1 \\ e_1 \; e_2 \; e_3 \; e_4 & 1 \\ a_1 \; a_2 \; a_3 \; a_4 & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \\ c_1 \; c_2 \; c_3 \; c_4 & 1 \end{vmatrix}}{D'} = Kad \qquad \frac{\begin{vmatrix} x_1 \; x_2 \; x_3 \; x_4 & 1 \\ e_1 \; e_2 \; e_3 \; e_4 & 1 \\ a_1 \; a_2 \; a_3 \; a_4 & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \\ c_1 \; c_2 \; c_3 \; c_4 & 1 \end{vmatrix}}{D},$$

$$\frac{\begin{vmatrix} x_1' x_2' x_3' x_4' & 1 \\ a_1 \; a_2 \; a_3 \; a_4 & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \\ c_1 \; c_2 \; c_3 \; c_4 & 1 \\ d_1 \; d_2 \; d_3 \; d_4 & 1 \end{vmatrix}}{D'} = Kae \qquad \frac{\begin{vmatrix} x_1 \; x_2 \; x_3 \; x_4 & 1 \\ a_1 \; a_2 \; a_3 \; a_4 & 1 \\ b_1 \; b_2 \; b_3 \; b_4 & 1 \\ c_1 \; c_2 \; c_3 \; c_4 & 1 \\ d_1 \; d_2 \; d_3 \; d_4 & 1 \end{vmatrix}}{D}.$$

We have four linear equations to determine $x'_1 x'_2 x'_3 x'_4$, which is the proper number to give us a unique solution. Moreover, we have the twenty-four essential parameters of our collineation, namely, the twenty coordinates of the invariant points and the four independent cross-ratios $K_{ab} K_{ac} K_{ad} K_{ae}$.

That there are only four independent cross-ratios is easily seen from the theorem that the product of cross-ratios around any invariant triangle is unity.

So the cross-ratio along some other line of the figure than the four chosen, K_{bd}, say, is directly dependent on K_{ab} and K_{ad}.

The system of four equations, just deduced, involving the variables as shown, is called the implicit normal form of the collineation.

We wish to express the four x''s explicitly in terms of x's and the natural parameters. This necessitates a solution of the four equations simultaneously, which by the ordinary methods of elimination is very cumbersome. But since the answer is known from the form which the collineation takes in space of lower order, it is sufficient to set up our explicit normal form and identify the resulting collineation with the implicit form that we are solving.

This form is:

$$(6) \qquad X'_i = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 & o \\ a_1 & a_2 & a_3 & a_4 & 1 & a_i \\ b_1 & b_2 & b_3 & b_4 & 1 & kb_i \\ c_1 & c_2 & c_3 & c_4 & 1 & k'c_i \\ d_1 & d_2 & d_3 & d_4 & 1 & k''d_i \\ e_1 & e_2 & e_3 & e_4 & 1 & k'''e_i \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & 1 & o \\ a_1 & a_2 & a_3 & a_4 & 1 & 1 \\ b_1 & b_2 & b_3 & b_4 & 1 & k \\ c_1 & c_2 & c_3 & c_4 & 1 & k' \\ d_1 & d_2 & d_3 & d_4 & 1 & k'' \\ e_1 & e_2 & e_3 & e_4 & 1 & k''' \end{vmatrix}} \qquad i = 1, 2, 3, 4$$

where k, k', k'' and k''' are K_{ab}, K_{ac}, K_{ad} and K_{ae} respectively.

In homogeneous form this becomes

$$(7) \qquad \rho x'_i = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & o \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_i \\ b_1 & b_2 & b_3 & b_4 & b_5 & kb_i \\ c_1 & c_2 & c_3 & c_4 & c_5 & k'c_i \\ d_1 & d_2 & d_3 & d_4 & d_5 & k''d_i \\ e_1 & e_2 & e_3 & e_4 & e_5 & k'''e_i \end{vmatrix} \qquad i = 1, 2, 3, 4, 5$$

To show that (5) and (7) are identical collineations, make (5) also homogeneous and take for the invariant pentahedroid the frame of reference whose vertices are

$$\begin{array}{lccccc} a-o & o & o & o & 1 \\ b-1 & o & o & o & o \\ c-o & 1 & o & o & o \\ d-o & o & 1 & o & o \\ e-o & o & o & 1 & o \end{array}$$

Then both (5) and (7) reduce to the same canonic form:

$$(8) \qquad \begin{array}{l} \rho\, x_1' = k\ x_1 \\ \rho\, x_2' = k'\, x_2 \\ \rho\, x_3' = k''\, x_3 \\ \rho\, x_4' = k'''\, x_4 \\ \rho\, x_5' = \ \ x_5 \end{array},$$

and the two collineations must be identical.

§ 8. The determinant of the form (7) consists of twenty-five terms, each term being a determinant of the fifth order, and factors easily into

$$D = k\,k'\,k''\,k''' \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ B_1 & B_2 & B_3 & B_4 & B_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 \\ D_1 & D_2 & D_3 & D_4 & D_5 \\ E_1 & E_2 & E_3 & E_4 & E_5 \end{vmatrix},$$

where A_{ij} is the cofactor of a_{ij} in the first determinant of this product.

We can write this product in the following way without altering its value:

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 \\ A_4 & B_4 & C_4 & D_4 & E_4 \\ A_5 & B_5 & C_5 & D_5 & E_5 \end{vmatrix} \times \begin{vmatrix} 1 & o & o & o & o \\ o & k & o & o & o \\ o & o & k' & o & o \\ o & o & o & k'' & o \\ o & o & o & o & k''' \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}.$$

The matrix of Δ is equal to the product of these three matrices, and we have therefore factored our transformation, which we may call T_2, into the product of three transformations with matrices (A), (k) and (a). Now (A) is the inverse transformatian of (a). Hence we have $T_2 = T_1^{-1}\, K\, T_1$.

Therefore any collineation in normal form is the transform of its canonic form by a collineation whose coefficients are the coordinates of the invariant points of the original collineation.

§ 9. *Inverse of Normal Form.* A glance at (5) shows that the inverse of T is obtained by simply replacing x_i by x'_i and $k^{(i)}$ by $\frac{1}{k^{(i)}}$. Hence:

$$T^{-1}:\ \rho x_i = \begin{vmatrix} x'_1 & x'_2 & x'_3 & x'_4 & x'_5 & o \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_1 \\ b_1 & b_2 & b_3 & b_4 & b_5 & \frac{1}{k} b_1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & \frac{1}{k'} c_1 \\ d_1 & d_2 & d_3 & d_4 & d_5 & \frac{1}{k''} d_1 \\ e_1 & e_2 & e_3 & e_4 & e_5 & \frac{1}{k'''} e_1 \end{vmatrix}.$$

§ 10. *Resultant of Two Collineations.* Let T and T_1 be two collineations with matrices (a) and (a'), respectively.

$$T:\ \rho x'_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 + a_{i5}x_5,$$

$$T_1:\ \rho' x''_i = a'_{i1}x'_1 + a'_{i2}x'_2 + a'_{i3}x'_3 + a'_{i4}x'_4 + a'_{i5}x'_5.$$

We can determine $T_2 = T\,T_1$ as follows:

$$T^{-1} = \tfrac{1}{\rho'}\triangle x_i = A_{1i}x'_1 + A_{2i}x'_2 + A_{3i}x'_3 + A_{4i}x'_4 + A_{5i}x'_5.$$

Now take the five equations of T^{-1} with one equation of T_1 and eliminate the x''s. The eliminant is

$$\begin{vmatrix} -\rho' x''_i & a'_{i1} & a'_{i2} & a'_{i3} & a'_{i4} & a'_{i5} \\ -\frac{\triangle}{\rho'} x_1 & A_{11} & A_{21} & A_{31} & A_{41} & A_{51} \\ -\frac{\triangle}{\rho'} x_2 & A_{12} & A_{22} & A_{32} & A_{42} & A_{52} \\ -\frac{\triangle}{\rho'} x_3 & A_{13} & A_{23} & A_{33} & A_{43} & A_{53} \\ -\frac{\triangle}{\rho'} x_4 & A_{15} & A_{24} & A_{34} & A_{44} & A_{54} \\ -\frac{\triangle}{\rho'} x_5 & A_{15} & A_{25} & A_{35} & A_{45} & A_{55} \end{vmatrix} = o$$

which solved for x''_i gives

$$\rho'' x''_1 = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & o \\ A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & a'_{j1} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & a'_{j2} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & a'_{j3} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & a'_{j4} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & a'_{j5} \end{vmatrix}$$

This is evidently T_2, the resultant of T and T_1.

§ 11. *Cross-ratio of the Resultant.* The form of the cross-ratio of the resultant of two collineations can easily be shown by considering the canonic forms of the collineations.

Suppose T and T_1 in canonic form are

$$T : \rho\, x_i' = k^{(i)}\, x_i ,$$
$$T_1 : \rho\, x_i'' = k_1^{(i)}\, x_i' .$$

By eliminating x', we have

$$T_2 : \rho\, x_i'' = k_1^{(i)}\, k^{(i)}\, x_i ,$$

that is to say: the cross-ratio of the resultant of two collineations along a line invariant under both collineations is equal to the product of the corresponding cross-ratios of the two collineations.

§ 12. *Roots of the Characteristic Equation.* It was shown in § 5 that there are five values of ρ for which a point is left invariant under T. Our normal form of T gives an easy solution for these five roots of the characteristic equation, showing their relation to the cross-ratios and the invariant points. The five roots must satisfy identities of the following form, where ρ_a is the value of ρ for which a point a is left invariant.

$$\rho_a a_i = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & o \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_i \\ b_1 & b_2 & b_3 & b_4 & b_5 & k & b_i \\ c_1 & c_2 & c_2 & c_4 & c_5 & k' & c_i \\ d_1 & d_2 & d_3 & d_4 & d_5 & k''d_i \\ e_1 & e_2 & e_3 & e_4 & e_5 & k''' & e_i \end{vmatrix},$$

$$\therefore \rho_a = \Delta' \quad \text{(where } \Delta' \text{ is the determinant of the invariant points.)}$$
$$\rho_b = k\,\Delta'$$
$$\rho_c = k'\,\Delta'$$
$$\rho_d = k''\,\Delta'$$
$$\rho_e = k'''\,\Delta'.$$

Evidently the ρ's are proportional to the cross-ratios of the collineation.

Part IV.

Types of Collineations.

§ 13. Collineations are classified by types according to the geometrical figure that they leave invariant. These figures can be deduced from a consideration of the character of the solution of the characteristic equation:

$$\begin{vmatrix} (a_{11}-\rho') & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22}-\rho' & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33}-\rho' & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44}-\rho' & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}-\rho' \end{vmatrix} = 0.$$

We shall consider first the solutions where our matrix is of rank four.

There are five roots which may be:

Five distinct roots.
Three single roots, one double root.
One single root, two double roots.
One triple root, one double root.
One triple root, two single roots.
One quadruple root, one single root.
One quintuple root.

Multiple roots show multiple points for the invariant figure. What sort of a configuration do we have at such a point?

Let two points a and b approach coincidence along a curve on a hyper-quadric surface with no singularities along the path of approach.

When b is near to a the coordinates of the line ab become

$$\begin{vmatrix} a_i & a_j \\ a_i + \triangle a_i & a_j + \triangle a_j \end{vmatrix}$$

$$= \begin{vmatrix} a_i & a_j \\ \triangle a_i & \triangle a_j \end{vmatrix}$$

$$= \rho \begin{vmatrix} a_i & a_j \\ \dfrac{\triangle a_i}{\triangle s} & \dfrac{\triangle a_j}{\triangle s} \end{vmatrix}$$

where $\triangle_{ai}$ indicates an increment of a_i.

At the limit these homogeneous coordinates become:

$$\rho \begin{vmatrix} a_i & a_j \\ \dfrac{da_i}{ds} & \dfrac{da_j}{ds} \end{vmatrix}.$$

But these are the coordinates of a line through two consecutive points of the curve; that is, a tangent to the curve at a. So that at a double point we have an invariant lineal element.

Similar considerations show that at a triple point we have an invariant plane having only this point in common with the rest of the invariant figure. And at a quadruple point we have an invariant space having only this point in common with the remainder of the invariant configuration.

What happens when the rank of this matrix is less than four?

In the first place, if the rank of the matrix is less than four for some value of ρ, ρ_1 say, we know that ρ_1 is at least a double root, for if the first minors of $\Delta(\rho_1)$ become o, $\Delta'(\rho_1) = o$, which is the usual criterion for a multiple root. Should the rank be lower than three, $\Delta''(\rho_1) = o$, and we have a root of at least multiplicity three; and so on.

What are our invariant figures in these cases? For the case of a matrix of rank three, the solution of three of the equations (3) of section 1 gives us the point or

points corresponding to the value of ρ, for which our matrix has this rank. In the solution of these equations, we may assign values arbitrarily to any two of the variables and determine the other three uniquely. Hence we get a single infinity of solutions for the ratios

$$x_1 \quad x_2 \quad x_3 \quad x_4$$
$$x_5' \quad x_5' \quad x_5' \quad x_5'$$

and our invariant figure corresponding to this value of ρ is a line of all invariant points, the line of intersection of the three spaces given by the three equations solved.

For a root for which the matrix is of rank 2 we have two of the equations to solve, and get a plane of invariant points.

If the matrix is of rank o all points are invariant under T, and we have the identical collineation. In deducing the complete invariant figure we are aided by the principle of duality. The figure must be completely dualistic with regard to points and spaces, and with respect to lines and planes.

Also, we know that the collineation in any plane of the figure must be one of the five well-known types of plane collineation. The considerations given above are sufficient to determine completely the invariant figure in every case except when we have a quintuple root for which the matrix is of rank 2 or 3. The last of these gives us three distinct types. We have a range of points in a line, hence, dualistically, a pencil of spaces through a plane. This plane may intersect the plane which is the dual of the line of points (1) in a point, (2) in a line, (3) or may coincide with it.

If the matrix is of rank 2 the line of axis of the two-dimensional space pencil may intersect the plane of all invariant points or may lie in the plane.

There are 27 types of collineation in the R_4, including the identical collineation, and following is a table in which the invariant elements are counted for each

type. In the case where one element of a range or pencil is specially singled out by some property, as for instance, when one point of a range is the vertex of a pencil of lines, it will be counted separate from the infinity of elements to which it belongs.

The first seven types occur when the matrix is of rank 4.

I. 5 points — all distinct.
10 planes.
10 lines.
5 spaces.

II. 4 points — 1 double, 3 single.
7 lines.
7 planes.
4 spaces.

III. 3 points — 2 double, 1 single.
5 lines.
5 planes.
3 spaces.

IV. 2 points — 1 double, 1 triple.
3 lines.
3 planes.
2 spaces.

V. 3 points — 1 triple, 2 single.
4 lines.
4 planes.
3 spaces.

VI. 2 points — 1 quadruple, 1 single.
2 lines.
2 planes.
2 spaces.

VII. 1 point — quintuple.
1 line.
1 plane.
1 space.

VIII. 3 single roots — 1 double, matrix of rank 3.
$3 + \infty$ points — 3 single points, 1 range of points.
$4 + 3\infty$ lines — 1 line containing range of points; 3 lines, axe of pencils of planes; 3 pencils of lines.
$4 + 3\infty$ planes — 1 plane, axis of space pencil; 3 planes containing pencils of lines; 3 pencils of planes.
$3 + \infty$ spaces — 3 single spaces; 1 pencil of spaces.

IX. 2 double roots, rank 3; 1 single root.
$1 + 2\infty$ points — 1 single point; 2 ranges of points.
$2 + 2\infty + \infty^2$ lines — 2 lines containing ranges of points; 2 pencils $+ \infty$ pencils of lines.
$2 + 2\infty + \infty^2$ planes — 2 planes, axes of pencils of spaces; $2 + \infty$ pencils of planes.
$1 + 2\infty$ spaces — 1 single space; 2 pencils of spaces, through two planes.

X. 1 double root, rank 3; 1 of rank 4; 1 single root.

$2+\infty$ points—1 single; 1 double; 1 range.

$3+2\infty$ lines—2 lines, axes of pencils of planes; 1 line, range of points; 2 pencils of lines.

$3+2\infty$ planes—2 planes containing pencils of lines; 1 plane, axis of space pencil; 2 pencils of planes.

$2+\infty$ spaces—2 single spaces; 1 pencil of spaces.

XI. 1 double root, rank 3; 1 triple root, rank 4.

$1+\infty$ points—1 triple point; 1 range.

$2+\infty$ lines—1 line, range of points; 1 line, axis of pencil of planes; 1 pencil of lines.

$2+\infty$ planes—1 plane, axis of pencil of spaces; 1 plane contains pencil of lines; 1 pencil of planes.

$1+\infty$ spaces—1 triple space; 1 pencil of spaces.

XII. 1 double root, rank 4; 1 triple root, rank 3.

$1+\infty$ points—1 double point; 1 range.

$2+2\infty$ lines—1 line, axis of pencil of planes; 1 line, axis of pencil of planes, and containing range of points; 2 pencils of lines.

$2+2\infty$ planes—1 plane, axis of pencil of spaces, and containing pencil of lines; 1 plane containing pencil of lines; 2 pencils of planes.

$1+\infty$ spaces—1 space; 1 pencil of spaces.

XIII. 1 double root, rank 4; 1 triple root, rank 2.

$1+\infty^2$ points—1 double point; 1 plane of points.

$1+2\infty^2$ planes—1 line, axis of pencil of planes; ∞^2 lines from point to plane, each containing pencil of planes; ∞^2 lines in a plane.

$1+2\infty^2$ planes—1 plane of ∞^2 points; ∞^2 planes through a line and points in plane; ∞^2 planes through lines from point to plane, each containing a pencil of lines.

$1+\infty^2$ spaces—1 double space; ∞^2 spaces through line.

XIV. 1 double root, rank 3; 1 triple root, rank 2.

$\infty+\infty^2$ points—1 plane of points; 1 range.

$\infty^3+\infty^2$ lines—∞^3 lines from range to plane; ∞^2 lines in plane of points.

$\infty^2+\infty_3$ planes—∞^2 planes from range to points in plane; ∞^3 planes through lines from range to plane.

$\infty+\infty^2$ spaces—∞ spaces through plane of points; ∞^2 spaces through range of points.

XV. 1 triple root, rank 3; 2 single roots.

$3+\infty$ points—2 single points, 1 point of range, and range.

$4+3\infty$ lines—3 lines, axes of pencils of planes; 1 line, range of points; 3 pencils of lines.

$4+3\infty$ planes—3 planes containing pencils of lines; 1 plane, axis of pencil of spaces; 3 pencils of planes; $3+\infty$ spaces.

XVI. 1 triple root, rank 2; 3 single roots.

$2+\infty^2$ points—2 single points; 1 plane of points.

$1+\infty^2+2\infty^2$ lines—1 line, axis of pencil of planes and of pencil of spaces; ∞^2 lines in plane; $2\infty^2$ lines from 2 points to plane of points.

$1+\infty^2+2\infty^2$ planes—1 plane of ∞^2 points; ∞^2 planes from line to points in plane of points; $2\infty^2$ planes through lines from 2 points to plane of points.

$2+\infty^2$ spaces—2 single spaces; ∞^2 spaces through a line joining 2 single points.

XVII. Triple root, rank 3; double root, rank 3.

$1+2\infty$ points — 2 ranges, and 1 point of 1.

$2+2\infty+\infty^2$ lines — 2 lines, ranges of points; 2∞ lines in 2 planes; axes of pencils of spaces; ∞^2 lines joining two ranges of points.

$2+2\infty+\infty^2$ planes — 2 planes; axes of pencils of spaces; 2∞ planes through ranges of points; ∞^2 planes through lines of 1 pencil of lines.

$1+2\infty$ spaces — 1 double space; 1 pencil of spaces.

XVIII. Quadruple root, rank 3; single root.

$1+\infty$ points — 1 single point; 1 range.

$1+\infty+\infty^2$ lines — 1 range of points; ∞ lines from point to range; ∞^2 lines containing ∞ pencils of planes.

$1+\infty+\infty^2$ planes — 1 plane from point to range; ∞ planes through range; ∞^2 planes consisting of ∞ pencil of planes.

$1+\infty$ spaces — 1 single space, and a pencil of spaces through plane from point to range.

XIX. Quadruple root, rank 2; 1 single root.

$1+\infty^2$ points — 1 single point; 1 plane of points.

$1+\infty^2+\infty^2$ lines — 1 line, axis of ∞^2 spaces; ∞^2 lines from point to plane of points; ∞^2 lines in plane of points.

$1+\infty^2+\infty^2$ planes — 1 plane containing ∞^2 points; ∞^2 planes through line axis; ∞^2 planes from point to lines in plane of points.

$1+\infty^2$ spaces — 1 single space; 1 pencil of ∞^2 spaces through line axis of plane.

XX. Quadruple root, rank 1; 1 single root.

$1+\infty^3$ points — 1 single point; 1 space of points.

$\infty^3+\infty^4$ lines — ∞^3 lines from point to space; ∞^4 lines in a space.

$\infty^3+\infty^4$ planes — ∞^4 planes through a point and lines of a space; ∞^3 planes in a space.

$1+\infty^3$ spaces — 1 single space; ∞^3 spaces through a point.

XXI. Quintuple root, rank 3.

$2+\infty$ points — range; and 2 points of range.

$3+2\infty$ lines — 2 lines containing pencils of planes; 1 line, range of points; 2 pencils of lines in 2 planes.

$3+2\infty$ planes — 2 planes containing pencils of lines; 1 plane, axis of pencil of spaces; 2 pencils of planes through 2 lines.

$2+\infty$ spaces — ∞ spaces through a plane; and 2 spaces containing pencils of planes.

XXII. Quintuple root, rank 3.

1 range of points, and 1 point of same.

$2+\infty$ lines — 1 line, axis of pencil of planes; 1 line, range of points; 1 pencil of lines.

$2+\infty$ planes — 1 plane, axis of pencil of spaces; 1 plane containing a pencil of lines; 1 pencil of planes.

1 pencil of spaces and 1 space of same, containing pencil of planes.

XXIII. Quintuple root, rank 3.

1 range of points.

$1 + \infty + \infty^2$ lines—1 line, range of points; each point of this range is the vertex of a plane pencil of lines. Each line of 1 pencil is the axis of a pencil of planes.

$1 + \infty + \infty^2$ planes—1 plane, axis of pencil of spaces; 1 pencil of planes through line, range of points; ∞^2 planes through the lines of one pencil of lines; 1 pencil of spaces.

XXIV. Quintuple root, rank 2.

$1 + \infty^2$ points—∞^2 points in plane and 1 point of same.

$1 + \infty^2 + \infty^3$ lines—1 line, axis of pencil of planes; ∞^2 lines in plane; ∞^3 lines from point where line cuts plane.

$1 + \infty^2 + \infty^3$ planes 1 plane of points; ∞_2 planes through line; ∞^3 planes through lines in plane and lines radiating from point.

$1 + \infty^2$ spaces.

XXV. Quintuple root, rank 2—same as XXIV, except that line of axis of ∞^2 planes lies in the plane of points.

XXVI. Quintuple root, rank 1.

$1 + \infty^3$ points—1 space of points and 1 point of same.

$\infty^4 + \infty^3$ lines—∞^4 lines in the space, ∞^3 lines radiating from point in the space.

$\infty^4 + \infty^3$ planes—∞^4 planes determined by lines radiating from the space and lines in the space; ∞ planes in the space.

$1 + \infty^3$ spaces.

XXVII. Matrix, rank 0.

Identical collineation leaving all points of the R_4 invariant.

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 15 — June, 1913.

(Whole Series, Vol. XVII, No. 15.)

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PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter

4–5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 15] JUNE, 1913. [WHOLE SERIES VOL. XVII, No 15

HISTOLOGY OF *SALSOLA KALI* L. VAR. *TENUIFOLIA* G. F. W. MEY.

BY MYRTLE GREENFIELD.

Plates XXXVI-XLII.

(Contribution from the Botanical Laboratory.)

(Submitted in partial fulfillment of the requirements for the degree of master of arts.)

SALSOLA KALI L. var. *tenuifolia* G. F. W. Mey, commonly known as the Russian thistle, was imported from eastern Europe to South Dakota in 1873 or 1874. It is well adapted to flourish in dry and unfavorable soil, and is widely distributed in such places. The material with which I worked was collected in Logan county, Kansas, during the summer of 1911, by F. A. Agrelius and Orville Wilson.

Salsola kali L. var. *tenuifolia* is not a thistle, as it is commonly called. It resembles very closely the common tumbleweed, *Amaranthus albus,* and also another tumbleweed, *Cycloloma atriplicifolium,* which belongs to the same family. The young and old plants differ widely. Both have terete leaves, but those of the young plants are long and succulent, while those of the old plant are short. These short leaves on the old plant are in rosette-like bunches. Each leaf, be it ever so short, produces a spine a trifle longer than those produced by the younger leaves. This added number of leaves, each with a spine, gives the plant a very prickly appearance. The photograph in figure 36, made by L. M. Peace, shows very clearly the difference between the old and young plants.

Two characteristics make the plant less troublesome than some of our other weeds. First, it is easily killed; and second,

Received for publication April 24, 1913.

since the seed coats are readily penetrated by water, the seeds germinate quickly and are not apt to live long in the soil. I found that upon soaking the seeds over night, the embryos, which are green, had broken the seed coats and were uncoiled. Dry seeds planted under artificial garden conditions were germinated and through the soil in three days. This speedy germination is characteristic of the seeds of xerophytic plants.

According to G. E. Patrick, Bulletin No. 26, Iowa Experiment Station, the Russian thistle is richer in protein than red clover; and I find, upon comparison with other data, that it is as rich in protein as alfalfa. The clovers obtain most of their nitrogen from the air, while the thistle derives its from the soil.

The most striking anatomical feature in this species is the anomalous structure of the stem. This anomaly occurs in all Chenopodiaceæ in which the growth in thickness is considerable (Solereder, 1908). It consists in the appearance of pericyclic strips of cambium, which originate and lose their activity successively, forming secondary bundles outward and conjunctive tissue. In other members of the family the pericyclic cambium is a continuous ring, forming concentric rings of wood and phloëm (Solereder, 1908). Figures 1 and 2.show the anomaly in *Salsola kali* L. var. *tenuifolia.* Figure 2 is a cross section of a young stem, where secondary xylem is represented by heavy stippling in the vascular bundle. Figure 1 is a cross section of a much older stem. Here the stippled sections in the vascular bundle ring indicate the masses of phloëm. These retain their cellulose walls, while they are surrounded successively by xylem and prosenchymatous conjunctive tissue with lignified walls. Figures 4 and 5 illustrate the same in more detail. The root exhibits the same anomaly of structure as the stem. (Fig. 3.) The same peculiarity of structure is common in Amaranthaceæ and Nyctaginaceæ (Solereder).

When viewed macroscopically, the stem of *Salsola kali* var. *tenuifolia* has usually eight principal longitudinal ribs more prominent than the channels and lighter in color. As the stem grows older these ribs and the trichomes that appear upon them become red. The ribs are composed of collenchyma cells with very thick cellulose walls. Figure 8 shows these cells in cross section, and figure 9 the same in longitudinal

section. These ribs in a stem five millimeters in diameter are 0.216 millimeters thick radially, and 0.29 millimeters thick tangentially. The lumen of the average collenchyma cell averages 0.0175 millimeters radially and tangentially, and 0.65 to 0.34 millimeters longitudinally. The walls average 0.0044 millimeters thick. The epidermal cells above these collenchyma cells are elongated, and there are no stomata here. (Fig. 13.) It is on these ridges that the trichomes appear. They consist of clothing hairs, some 0.2 millimeters long, others so short that they are nearly knobs. All of them have small basal cells. In some cases the basal cells are very small. (Fig. 11.) In all cases the cuticle of the trichomes is rough. Since the trichomes are on the ribs, they overhang the channels to some degree and quite effectively protect that portion of the stem which has the stomata.

Between the ribs the epidermal cells are more or less isodiametric, and the stomata are frequent. Their guard cells are arranged transversely to the longitudinal axis. The guard cells have small projections of the cuticle that lap together when the stoma is closed. The lumens of the epidermal cells average 0.07 millimeters tangentially and 0.0175 millimeters radially. The upper wall averages 0.0175 millimeters thick, 0.0058 millimeters of this being cuticle. The lower wall averages 0.0029 millimeters thick.

Beneath the epidermal cells in the channels are palisade parenchyma cells averaging 0.017 millimeters tangentially and 0.0525 millimeters radially. Their walls average 0.0014 millimeters thick. Beneath these palisade parenchyma cells there is a second row of chloroplast-bearing cells with the plastids massed on the inner side of the cells. These cells average 0.05 millimeters tangentially and 0.0175 millimeters radially and longitudinally. The upper walls average 0.0026 millimeters thick and the lower 0.00286 millimeters thick. Figure 5 shows these cells as they occur in the stem. In figures 27 and 28 the corresponding and very similar cells in the leaf are shown in detail.

Beneath the cells just described and the subepidermal collenchyma rib is a strongly developed aqueous tissue, consisting of parechyma cells averaging 0.0875 millimeters radially, 0.14 millimeters tangentially, and 0.13 longitudinally. The walls average 0.0058 millimeters thick. The tissue is two or

three cells deep. The cells have a few scattered chloroplasts, and many of them have large clusters of calcium oxalate crystals. (Figs. 5, 9, 14.) Beneath this aqueous tissue or parenchyma of the cortex is a starch sheath. The lumen of the cells averages 0.0525 millimeters tangentially, 0.0175 millimeters radially, and 0.105 millimeters longitudianally. The walls average 0.0058 millimeters thick.

Beneath the starch sheath is the parenchyma of the pericycle, consisting of one row of cells with walls 0.0018 millimeters thick and the lumen averaging 0.0262 millimeters tangentially, 0.0087 radially, and 0.122 millimeters longitudinally. The starch grains in the starch sheath are compound. (Figs. 16 and 17.)

The anomaly of the stem structure gives the cross section of the vascular bundle ring a peculiar appearance. The small mass of phloëm cells with their cellulose walls are opposite the tracheal vessels. True spiral tracheal tubes occur in the primary bundle. In the secondary bundles, pitted and spiral tracheids predominate, with some pitted and scalariform tracheal tubes. (Figs. 20 and 21.)

The phloëm consists of undivided mother cells of sieve tubes and companion cells and phloëm parenchyma cells. Apparently there are no true sieve tubes with their sieve plates. The average phloëm cell has walls 0.00147 millimeters thick, with a lumen 0.0088 millimeters radially and tangentially, and 0.105 millimeters longitudinally.

In the phloëm a great many of the cells are filled with a food product giving the test for protein, which it probably is. Occurrence of the stuff is common in the phloëm of the root. Frequently a cell in the phloëm of the stem is filled with calcium oxalate crystals cubical in shape. (Fig. 19.)

Surrounding the tracheal tubes in the primary bundle are xylem parenchyma cells with plain pits. In this bundle the parenchyma cells occupy a considerable space. In the secondary portion of the vascular bundle only a few xylem parenchyma cells occur, and these immediately next to the tracheal tissue. Surrounding these cells, and occupying the space between the groups of tracheal tubes, are wood fibers, shown from mascerated sections in figure 15.

The xylem parenchyma cells in the primary bundle have walls 0.0021 millimeters thick. The lumen is 0.043 millime-

ters radially and tangentially, and is from 0.105 to 0.21 millimeters longitudinally. The average tracheæ have walls 0.00437 millimeters thick. The lumen is 0.0437 millimeters radially. The tracheids are often irregular in shape. (Fig. 21.) Their walls are about the thickness of the walls of the tracheæ, and their average length is 0.13 millimeters.

Medullary rays are not found in the old stem of *Salsola kali* var. *tenuifolia,* nor, according to Solereder (1908), are they found in the stem of any species of the family. In families which have similar anomalies medullary rays are found in a number of species.

The pith is composed of exceedingly large, pitted cells averaging 0.216 millimeters in diameter. The walls are 0.0086 millimeters thick. In a great many of the cells are calcium oxalate crystals, not clustered as those in the aqueous tissue, but scattered in the cells.

In a stem 1.36 millimeters in diameter 2.8 per cent of the tissue is devoted to water transportation.

THE LEAF.

The leaf of *Salsola kali* var. *tenuifolia* is narrow, terete and succulent. The epidermis consists of a single layer of cells. The lumens of these cells average 0.0175 millimeters radially and 0.065 millimeters tangentially. The outer wall is 0.0058 millimeters thick, 0.0019 millimeters of this being cuticle. The inner wall is 0.00284 millimeters thick. The stomatal openings are arranged parallel with each other and transversely to the median vein of the leaf. The hairy covering is very similar to the hairy covering of the stem. The trichomes near the base are longer than those near the tip of the leaf. Unicellular trichomes are not found, but uniseriate trichomes are frequent.

Immediately beneath the epidermis is a layer of palisade parenchyma cells. (Figs. 25, 30, and 31.) Figure 27 shows these cells in detail. The lumens of these cells average 0.0525 millimeters radially and 0.0172 millimeters tangentially. The walls average 0.00144 millimeters thick. Below the palisade parenchyma cells is a row of short chloroplast-bearing cells with their chloroplasts massed toward the inner side of the cells. (Figs. 27 and 28.) The lumens of the cells average 0.0262 millimeters radially, 0.035 millimeters tangentially, and 0.0262 millimeters longitudinally. The upper wall averages

0.0027 millimeters thick and the lower averages 0.0041 millimeters thick.

The chief part of the internal tissue of the leaf is composed of aqueous or thin-walled parenchymatous tissue. This tissue encloses the median vein, together with its branches, lying immediately below the short chloroplast-bearing cells. The lumen of the aqueous tissue cells averages 0.2437 millimeters longitudinally and 0.0875 millimeters radially. The walls are 0.0058 millimeters thick. These cells have a few chloroplasts (fig. 29), and frequently calcium oxalate crystals. Their frequency is shown in figure 22.

Figure 30 shows the leaf in cross section. Figure 34 shows the details of the median vein in cross section, and figure 26 a part of the same in longitudinal section. The water-conducting tissue consists of tracheæ averaging 0.00875 millimeters in width, with annular thickenings. The phloëm parenchyma and undivided mother cells have walls averaging 0.00134 millimeters in thickness.

On the upper side of the leaf, from its connection to the stem, extending about one-third of the length of the leaf, is a depression (*a*, fig. 30). Below this the cylinders of palisade parenchyma and short chloroplast-bearing cells are broken, and through the gap the aqueous tissue extends to the epidermis, the cells of which in this region are elongated parallel with the long axis of the leaf (*a*, fig. 24).

On the under side of the leaf, near the tip, the palisade parenchyma cylinder and the cylinder composed of short chloroplast-bearing cells are again broken. Beneath the epidermis, which in this region has elongated cells, are lignified bast fibers. As the tip is neared, more of the leaf is devoted to these bast fibers, until the median vein is entirely surrounded by bast fibers. This forms the spine. Figure 31 is a cross section near the tip of the leaf, and figure 32 a cross section through the spine. Figures 30, 31 and 32 are all drawn to the same scale, so that the decrease in diameter as the leaf narrows into the spine is shown. Figure 25 shows the tip of the leaf longitudinally. The spine averages 1.3 millimeters long and 0.225 millimeters wide. The bast fibers average 0.26 millimeters long and have walls 0.005 millimeters thick.

The following list of the families having an anomalous stem

structure similar to that of the *Salsola kali* var. *tenuifolia* is compiled from Solereder's Systematic Anatomy of the Dicotyledons (1908):

DILLENIACEÆ.

The anomaly is present only in the genus *Poliocarpus*, and consists in the formation of successive concentric rings of bundles. The secondary cambium appears in the parenchymatous pericycle internally to the primary groups of bast fibers.

MENISPERMACEÆ.

Anomalous stem structure occurs only in certain members of the family, and in these cases a transverse section shows several concentric rings of vascular bundles, which either encircle one another completely on all sides of the transverse section or only develop strongly on one side. These secondary arcs or rings of vascular bundles are due to the activity of a cambium which arises in the innermost part of the primary cortex.

CAPPARIDEÆ.

Anomalous stem structure in the form of successive rings of growth has been met with in the genera *Mærua*, *Roydsia*, *Cadaba*, and *Forchhammeria*.

In *Mærua uniflora* secondary rings arise intrafascicularly in the zone of bast. (DeBary, 1877.)

In *Forchhammeria trifoliata* Radlk., *Mærua oblongifolia* A. Rich., and *Roydsia*, according to Radlkofer, they arise on the inner side of the sclerenchyma of the pericycle.

In *Forchhammeria apiocarpa* Radlk. and *Forchhammeria pallida* Liebm., according to Radlkofer, they arise on the outer side of the pericyclic sclerenchyma.

In *Cadaba glandulosa* the secondary rings arise in the outer primary cortex, according to Krüger (1884).

POLYGALEÆ.

Anomalous structure of the axis was first described by the older authors in *Securidaca volubilis* and *Comesperma*. According to H. Schenck (1893), we must add: *Bredemeyera*, *Schenckiana* Chodat, *Securidaca lanceolata* St. Hil., and *Securidaca sellowniana*, and, according to Chodat, *Moutabea guianensis*. The formation of the secondary bundles, according to H. Schenck, takes place in the parenchymatous pericycle on the inner side of the bundles of primary hard bast.

CARYOPHYLLACEÆ.

Successive rings of bundles appear in the root and stem of *Spergularia* and species of *Polycarpæa*. Secondary rings of growth have been observed by Morot in the root of *Lepigonum marinum*, by Peterson in the root of *Lipigonum salinum*, by H. Schenck in the root and stem of *Spergularia media* Presl. and by Solereder in the root of *Polycarpia fragilis* Del., and *P. corymbosa* Lam. They arise extrafascicularly.

HIPPOCRATEACEÆ.

According to Fritz Müller (1866), secondary rings of vascular bundles are found in the stems of the genus *Salacia*. These arise from a cambium developed in the pericycle.

AMPELIDACEÆ.

According to Schenck (1893), anomalous structure of the axis is known only in a single genus, *Tetrastigma*. Narrow strips of cambium are formed from the phloëm parenchyma.

LEGUMINOSÆ.

Subfamily PAPILIONOIDEÆ.

Successive rings of growth in the pericycle or in the primary cortex in species of *Wisteria*, *Mucuna*, *Spatholobus*, *Pueraria*, *Pachyrhizus*, *Rhynchosia*, *Dalbergia*, and *Machærium*.

LEGUMINOSÆ.

Subfamily CESALPINIOIDEÆ.

Secondary rings of vascular bundles take place in the pericyclic parenchyma on the inner side of the sclerenchymatous ring, according to H. Schenck (1898), in *Bauchinia rubiginosa* Bong. and in *B. angulosa* Vog., and, according to Brandis and Gamble, in *B. vahlii* W. & A.

FICOIDEÆ.

In the woody species rings or arcs of meristem appear successively in the pericycle of the stem, giving rise to secondary vascular bundles and intermediate tissue. The anomalous structure of the stem is usually repeated in the root.

UMBELLIFERÆ.

The development of an intrafascicular cambial ring has been observed by Möbius in *Eryngium lassauxii* Decne. and *E. serra* Cham. et Schlecht; externally it produces bast-like tissue, internally xylem with vessels.

RUBIACEÆ.

According to Witte (1886), cortical bundles originate in the bast of *Chiococca racemosa;* first a group of vessels is formed, around which a cambium ring appears, producing woody tissue with medullary rays internally and phloëm externally.

COMPOSITÆ.

In the genus *Mikania* a second zone of growth is situated on one side of the primary vascular bundle ring. This originates in the parenchyma of the pericycle. In *Kleinia spinulosa* Hort. Petrop., according to Joh. Müller (1893), there is a secondary vascular bundle ring arising from the phloëm of the primary ring. According to Kruch (1890), in the basal part of the stem of *Dahlia imperialis* a secondary ring arises from the endodermis.

CANDOLLEACEÆ.

According to Vesque (1878), a secondary ring of fiber-like sclerenchyma enclosing a few vessels and small phloëm groups is found in *Stylidium adnatum* R. Br., *Stylidium bulbiferum* Benth., *Stylidium*

cicatricosum Sond., *Stylidium dichotomum* D. C., *Stylidium fruticosum* R. Br., *Stylidium lancifolium, Stylidium rhynchocarpum* Sond. This ring is due to the activity of a secondary meristem appearing in the pericycle immediately below the endodermis (Van Tieghem, 1883-'84). The anomaly occurs in the roots of these also, and in the root but not the stem of *Stylidium graminifolium* and *S. spinulosum* Van Tieghem (1878).

PLUMBAGINACEÆ.

In *Acantholimon* the appearance of successive rings of growth is combined with the occurrence of the medullary vascular bundles. It has not been determined whether the secondary ring of bundles arises in the pericycle or in the bast portion of the primary vascular ring. In the genus *Ægialitis* the cross section appears similar to the cross section of the genus *Acantholimon,* but the meristem is from the layer of cells immediately outside the sclerenchymatous bands (Solereder, 1908).

PRIMULACEÆ.

In the group *Officinales* and in the *Auriculas* a system of vascular bundles is formed in the pericycle of the root stock.

CONVOLVULACEÆ.

A vascular bundle ring appears in *Argyreia, Calonyction, Ipomœa, Maripa, Merremia, Porana,* and *Rivea.* The ring is due to the activity of a secondary meristem which arises on the outer margin of the primary phloëm and internally to the bast fibers of the pericycle.

BIGNONIACEÆ.

A secondary zone of growth appears in the cortex of *Callichlamys* Miq., *Glaziovia* Bureau, and *Haplolophium* Cham.

VERBENACEÆ.

In the genus *Avicennia* successive rings of growth appear, all extrafascicular in origin.

PHYTOLACCACEÆ.

Some of the genera belonging to the tribes Rivineæ and Euphytolacceæ are distinguished by having an anomalous stem structure which consists in the appearance of successive rings of vascular bundles in the pericycle. This anomaly also occurs in some species of *Phytolacca,* also in *Ercilla volubilis* (Krüger, 1884), *Seguieria floribunda* Benth., *S. longifolia* Benth. (Solereder, 1908), and *S. americana* (Schulz).

NYCTAGINACEÆ.

Anomalous structure of the stem apparently occurs in all the woody species, and is also found in a certain number of the herbaceous species. It has been recorded by various authors (Regnault, 1860; Unger, 1840; Grönlund, 1872; Peterson, 1879-'80; DeBarry, 1877; Radlkofer, 1883; Solereder, 1885; H. Schenck, 1893; Houlbert, 1893) in numerous species of the genera *Mirabilis, Oxybaphus, Boerhavia, Bougainvillea, Pisonia, Eggersia, Phæoptillon, Neea, Leucaster,* and *Cryptocarpus.* The secondary meristem develops in the parenchymatous pericycle.

AMARANTACEÆ.

Anomalous stem structure occurs in species of the following genera: *Deeringia, Celosia, Hermbstædtia, Rodetia, Bosia, Chamissoa, Amarantus, Acnida, Pupalia, Psilotrichum, Aerua, Achyranthes Telanthera, Alternanthera, Gomphrena, Frœlichia,* and *Hebanthe.*

Like the Nyctaginaceæ, the secondary meristems develop in the parenchymatous pericycle.

LORANTHACEÆ.

Successive rings of growth appear in *Nuytsia* and *Strychnos.*

BUXACEÆ.

Secondary rings of bundles occur in *Simmondsia californica.* These secondary rings of bundles arise in the parenchymatous portion of the pericycle, internal to the pericyclic groups of sclerenchymatous fibers.

According to Solereder (1908), only four families other than Chenopodiaceæ have longitudinal ribs of collenchyma. These are Cruciferæ, Amarantaceæ, Labiatæ and Umbelliferæ.

The stomata of *Salsola kali* var. *tenuifolia* are so orientated that their pores are transversely arranged to the longitudinal axis of both leaf and stem. In the family Batidaceæ a majority of the stomata are similarly orientated. In the family Loranthaceæ, on the branches of *Antidaphne, Arceuthobium, Lepidoceras, Loranthus, Nuytsia,* and *Viscum,* a similar orientation is found. In some members of Santalaceæ the pores of the stomata are transverse to the longitudinal vein and in other members they are parallel to the vein.

Calligonum comosum L. Herit., a desert plant of the family Polygonaceæ, with advanced reduction of foliar organs, has a layer of palisade tissue in the primary cortex and a sheath of collecting cells below the palisade cells. There is also a development of aqueous tissue in the cortex. All these characteristics are found in *Salsola.*

METHODS.

All the drawings were made from sections cut in paraffin, with the exception of a few drawings made from macerated material, and the drawings 22 and 23 made from the whole leaf. The permanent sections were stained in safranin, but a number of the drawings and all of the measurements were made from sections stained with iodine.

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THE KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 16—June, 1913.

(Whole Series, Vol. XVII, No. 16.)

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THE ANATOMY OF GÆRTNERIA DELTOIDEA.....*Beryl Lovejoy.*

PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter.

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 16] JUNE, 1913. [WHOLE SERIES VOL. XVII, No 16

GÆRTNERIA DELTOIDEA TORR.

BY BERYL LOVEJOY.

Plates XLIII-LIV.

THIS study, carried on in the laboratories of the Botany Department of the University of Kansas, under the supervision of Professor Stevens, was made from material collected during the summer of 1908 by Mr. L. M. Peace, on the northern slope of Tumamoc Hill, near the Desert Laboratory at Tucson, Ariz. Here the soil is of a coarse, gravelly texture, overlying caliche; and *Gærtneria deltoidea,* in company with *Opuntia fulgida,* are mentioned by V. M. Spaulding, in his Distribution and Movement of Desert Plants, as constituting the greater part of the vegetation of this association, which is determined for the most part by soil relations.

In the collecting of *Gærtneria,* jars containing a 2 per cent solution of formalin were taken by Mr. Peace into the field, and the plants, as soon as gathered, were placed in these. Specimens of the root, stem and leaves were obtained at two different dates, the first at the beginning of the summer season of rain, July 31, a later collection being made August 3.

The genus *Gærtneria* is described by Britton and Brown as belonging to the family Ambrosiaceæ (Reichenb). Gray describes it as "*Franseria*" and places it among the Compositæ, as does J. D. Whitney and others.

The species *deltoidea* Torr. is described by J. D. Whitney (1880) as being a member of the Compositæ, "herbaceous, with more or less woody base, low, canescent, with a fine and

Received for publication May 24.

close wooliness, which is partly deciduous with age; branches slender, leaves varying from deltoid ovate or almost hastate to rhombic lanceolate, obtusely and finely serrate, sometimes sparingly incised, on slender petioles, with sterile heads rather loosely racemed, spines of the ovoid two-flowered involucre flat and thin, broadly lanceolate, subulate, pubescent, or almost glabrous."

In harmony with its environment, this plant has protective devices, limiting transpiration. The leaves are comparatively small, varying, in specimens studied, from 9 mm. to 29 mm. in length, and from 2 mm. to 16 mm. in width. They are seen to be hoary, with a gray pubescence, this thick covering of hairs minimizing the loss of water caused by its desert environment. Compensating the smallness of the leaves, the stem subepidermal tissues contain chloroplasts.

The leaves are borne by slender petioles on so much shortened branchlets that they seem many times almost to originate in one plane. These shortened branchlets are of approximately equal length, and are arranged alternately by thirds upon the older branches.

Gærtneria deltoidea is a low winter perennial, showing in all its phases a pronounced xerophytic structure. As just indicated, its greatest development occurs during the winter months, when it blooms and produces its fruits, which are bur-like, and are armed with several ranks of prickles or spines. These reach maturity and are ready to be dispersed by animals or other carriers early in April. The low diffuse habit of the plant, as a whole, is adapted to two main facts: first, the drying effect of high winds; and second, the intense heat and light of the desert environment.

In this study the drawings were made, for the most part, by means of the camera lucida and the projection lantern from sections and macerations of formalin material.

HISTOLOGY OF THE LEAF.

In presenting the histology of this part of the plant, the discussion of its cellular anatomy naturally falls under three general heads: first, the epidermal tissues; second, the fundamental tissues; and third, the vascular system.

The Epidermis.

The epidermal cells are variable in size. The inner tangential and radial walls are cellulose, the outer wall is cellulose

in its inner part, with a cutinized outer layer .001 mm. in thickness.

There are approximately 371 stomata per sq. mm. on either side of the leaf, nearly the average number found in plants, and there seem to be no essential differences in size and shape of the stomata on the two sides of the leaf. Projecting above from the side walls of the guard cells adjoining the air passages are the ridge-like protuberances or rims characteristic of guard cells. (Fig. 31.) These pore structures are highly cutinized and approximate when the stoma closes, thus materially reducing transpiration. The guard cells of *Gærtneria* are normally at the level of the general epidermis, rarely rising above it; but in the case of a leaf infected by a fungus, a curious modification of the epidermis tissues occurred, the outer tangential wall becoming highly thickened, and the guard cells being sunk below the surface. (See fig. 34.)

Two distinct varieties of hairs appear on the leaf, namely, elongate uniserial clothing, and short, two-ranked glandular hairs, these two forms making up the typical hairy covering of the Compositæ. (Solereder, 1908.)

The uniseriate clothing forms are much the more numerous on the leaf, being seen under the microscope as a thick mat on the leaf surface. (See fig. 32.) Their association in the epidermis is shown in figures 33 and 27, where they are seen to be outgrowths of single epidermal cells.

The short two-ranked hairs belong to the class of external secretory organs, widely distributed throughout the Compositæ. They are regularly biseriate, short-stalked or appearing as sessile, and have all walls entirely cutinized. They secrete a substance that has a golden-brown color in formalin solution. This substance from formalin material is soluble in ether and alcohol, and when dissolved out by this means there is left in each cell of the gland a nuclear body. These are relatively large and appear to be intimately associated with the secretion of the brown substance.

The Fundamental Leaf Tissue.

The fundamental or mesophyll tissue shows the typical leaf structure of palisade and spongy parenchyma, with the occasional occurrence of bast fibers and stone cells subtending the veins.

Perhaps the most interesting feature of the leaf is the great degree of variability shown in the distribution of the palisade and spongy parenchyma tissues. In one continuous cross section of the leaf may be seen, for a distance, palisade cells on the upper side, with spongy parenchyma below; then these conditions may be reversed for a space, the palisade being below and the spongy being above; again, no palisade tissue can be found on either side, or the palisade cells may extend the whole width of the mesophyll.

As shown in figure 39, *a*, *b*, and *c*, the spongy parenchyma has no fixed region in the leaf where it appears. This tissue averages three cells in depth from epidermis to vascular system, and is interspersed with many though usually not large intercellular spaces. The cell walls are slightly thicker than those of the palisade cells, and the cells, which are spheroidal, elongate or ovoid in shape, have an average cavity diameter of .0301 mm. The chloroplasts are relatively large and vary in number from three to eight.

The palisade cells have the characteristic cylindrical shape, are very thin-walled, and extend from one to five or six cells deep radially. Small but frequent intercellular spaces are found in connection with the palisade tissue. In figure 36 is shown a composite drawing through the leaf tangentially, showing a surface view of the palisade cells.

The number of chloroplasts in each cell averages twenty-one. They are relatively large in size, projecting far into the cell cavity. The following table, taken from Haberlandt's Physiologische Pflanzenanatomie, and showing the number of chloroplasts found in the mesophyll tissue of a number of plants, will give an idea of the relatively few chloroplasts in the leaf of *Gærtneria:*

Plant	Palisade	Spongy.
Pulmonaria officinalis	85	15
Ricinus communis	82	18
Brassica rapa	80	20
Galeopsis tetrahit	79	21
Trapæolum majus	77	23
Helianthus annuus	73	27
Phaseolus multiflorus	69	31

In the mesophyll tissue, associated with the veins are sclerenchymatous cells, ranging in shape from stone cells to typical bast fibers.

Water-storage Tissues.

That the leaf of *Gærtneria deltoidea* is unusually well supplied with water-storage tissue is shown by the fact that the ultimate ends of the veins vary from .2 to .025 mm. in distance apart from each other. A portion of the water used by the leaf passes straightway into the palisade cells from water tubes; other portions pass into large cells making up that part of the venation system used primarily in water storage. As there are chloroplasts found in these cells, it would appear that they function to some degree in photosynthesis as well as in water storage. In figure 43 are shown palisade cells lying in connection with water-storage parenchyma cells, thus directly gaining access to the stored water. In connection with the venation system of the leaf are also other water-storage cells, which, unlike the cells just mentioned, have highly thickened and plain-pitted walls, and in most cases these lie near the edge of the leaf at the ultimate ends of the vein branches there.

In studying the venation system of the leaf, it was found advantageous to put the leaf sections, etc., through a clearing process. This clearing of cells was very satisfactorily accomplished by using Mr. L. M. Peace's method of cell clearing and bleaching, which is described in *Plant World,* vol. 13, April, 1910.

By this method whole leaves were successfully cleared of cell contents, and then, when put in an alcoholic solution of safranin for a period of twenty-four hours, they showed with remarkable clearness the whole system of leaf venation, the tracheal elements being the only part of the leaf to take on, to any degree, the safranin stain. As shown in figure 30, the leaf of *Gærtneria* has a prominent midrib extending the full length of the leaf, with two lesser lateral longitudinal veins which are distinct for only a portion of the distance from the base to the tip of the leaf. From these extend branches throughout the leaf in every direction, the distances between the ultimate ends of which vary from .2 mm. to .025 mm.

Many of the veins end in the midst of a bundle of water-storage parenchyma cells, as described above. In a cross section through the petiole of the leaf near its base were found ninety-four tracheal tubes, the average diameter of which was .0055 mm.

The phloëm in the leaf vascular system is made up of undivided mother cells of sieve tubes, and cambiform cells. These have much the same appearance in size and shape, both varying from .0075 to .0087 mm. in cross diameter, and in length from .009 to .0182 mm. The average thickness of the cell wall is .00012 mm.

In the case of a leaf infected by a fungal growth there were found cells lying just without the phloëm, which showed a peculiar structure. They were sclerenchymatous in appearance, and when treated with chloriodide of zinc they were seen to have a secondary core of cellulose, these lying in a primary lignose matrix. (See fig. 45.)

THE STEM.

The stem is herbaceous, with fine pubescence at the growing apices, tending to become woody at from one to two inches back from the apex, and as it approaches the base of the shrub the stem covering becomes very rough, dark, and deeply seamed. The woody part of the plant is highly resistant to sectioning, hydrofluoric acid having little or no effect upon portions of the wood, even after continuous application for some four months.

The most interesting features of the stem are those seen in cross section, the fluctuation in extent and position of bast and vascular tissue being the most striking. This variability is shown in the series of drawings, figures 1, 2, and 3. The sections used for these representations were taken from a series cut in paraffin, young parts of the stem being taken a few centimeters back from the growing apex. In describing each tissue in detail there will be a definite order followed, beginning with the innermost one.

The Pith.

The pith occupies a somewhat pentagonal area; the individual cells are also pentagonal to circular in shape, and in the very young stem, within 3 mm. of the growing apex, have a cellulose cell wall approximately .0014 mm. in thickness. (Fig. 5.) At a distance from 1 to 3 cm. back from the growing apex, however, the walls take on a heavy secondary thickening and become highly lignified. The secondary thickening begins with the cells at the center of the pith area, and gradually extends outward until all the walls of the pith have taken on this thickening and lignification. In this latter con-

dition all the walls are deeply pitted with plain pits, and have much the appearance of stone cells. (Figs. 4 and 7.) In the youngest condition of these pith cells, before this secondary increase in thickness has begun, chloroplasts are present. (Fig. 6.) Relatively few are found in the central portion of the pith, but the chloroplasts become numerous in those cells lying at the outer edge of the pith area.

The Medullary Rays.

The medullary rays vary in different parts of the stem in frequency of occurrence and in extent of tissue. They are found in that part of the stem where the vascular system is made up of comparatively distinct, isolated bundles, but become inconspicuous and tend to disappear as the vascular bundles approach the position of a complete ring. In the very young stem chloroplasts are found scattered here and there in the medullary ray cells, showing that here, as in the pith, there is some photosynthetic action.

The Xylem.

The xylem of the stem varies from isolated bundles to a complete cylinder or ring around the stem. It is made up of wood parenchyma, wood fibers, fiber tracheids, and tracheal tubes. The wood fibers, in company with the tracheal tubes, make up the larger part of the xylem tissue. The fiber tracheids are fairly plentiful, and are of various lengths and widths, but are always much shorter than the wood fibers. The water tubes average 12 to 14 in the region of each vascular bundle. Along the tube, at intervals which average .058 mm. in length, appear the remains of the original cross-walls at the ends of tube elements, as they seem in some cases to lie in the plane of union of adjacent tubes. They are shown in figure 51.

The Phloëm.

The phloëm region, as that of the xylem, is variable in extent, being found as isolated portions, and also as a continuous ring, undivided even by medullary rays. The phloëm is composed of extremely small cells in cross section, these being phloëm parenchyma, sieve tubes, and companion cells. The sieve plates are thin and have perforations so minute as to be clearly distinguished only by the aid of the 2 mm. oil immersion lens. In the phloëm cells are found stored nitrogenous substances.

In young stems the vascular system is separated from the hard bast by a region of pericycle ground parenchyma containing chloroplasts, this part of the pericycle, in company with certain of the medullary ray cells and those of the pith, functioning in photosynthesis.

Bast Fibers.

Just without the thin-walled parenchyma of the pericycle lies the bast tissue, varying in extent from isolated bundles to a complete rigid cylinder around the stem. On comparing cross sections taken at different heights (compare figs. 1, 2, and 3), the bast appears to form a sort of anastomosing structure, dividing in places to allow for conduction of materials between the pericycle and cortex. The bast fibers are sparingly pitted, and have the usual long, tapering form.

Endodermis.

There is a very distinct endodermal tissue or starch sheath lying just without the bast ring, this being a type character of the Compositæ (Solereder), heavily loaded with large starch grains, these forming a dark purple ring around the stem, when it is sectioned and treated with iodine.

Lying in this endodermal region are found secreting spaces, occurring usually in pairs at the angle where the bast ring alternately bulges out and becomes depressed, there often being twenty of these spaces seen in cross section. These endodermal secreting canals (fig. 22), forming the type characteristic of the order Compositæ (Solereder, 1908), are in *Gærtneria* not evident much beyond 3 cm. back from the growing apex.

Cortex.

The cortex is made up of cortex parenchyma and collenchymatous tissue. The thin-walled parenchyma of the cortex averages six cells deep, and is made up of ovoid or cuboidal cells, with cellulose walls. There are many, though inconspicuous, intercellular spaces in this region, and, until the tissue becomes transformed into cork, which occurs later by the suberization of its walls, it has an active part in food manufacture.

There is found in this region no definite cork cambium, but, as stated above, the walls of the cortex cells finally become suberized. Solereder (1908, vol. I, p. 493) reports the absence in a somewhat similar case of a distinct cork cambium in

Styphelia læta, of the family Epacrideæ. The collenchymatous cells, lying two to three layers deep, are of the typical sort, and also contain chloroplasts in sufficient numbers to be of significance in photosynthesis.

Epidermis.

The epidermis of the stem, as that of the leaf, is beset with clothing and glandular hairs. Here, however, the short glandular hairs are much more numerous than on the leaf. They each have the same general structure as those on the leaf, hence no discussion of that is necessary here.

The epidermis of the stem occurs as a single layer of cells, elongated longitudinally, and having, in the very young stem, walls of cellulose. As the stem becomes old the walls thicken, and there is produced a cutin layer.

A curious exfoliating process, whereby the cambium forms cork and chlorenchyma rings, which replace the secondary pericycle chlorophyllous tissue, just as the latter replaced the first cortex chlorophyll tissue, is reported by Cannon, W. C. (1908), as peculiar to *Gærtneria*, and to be repeated several times until in the oldest parts of the branch the portions cut off and those reformed no longer contain chlorophyll.

THE ROOT.

The root system of *Gærtneria* (in keeping with characteristic xerophytic structure) is well developed. The older roots are seen to have a deeply furrowed, rough, coarse outer covering, and are so very hard as to permit of no thin sectioning, even after having been disilicified for some weeks. In cross section of the very young root appears a central xylem cylinder, which is highly lignified and surrounded by a phloëm ring, which in its turn is surrounded by a pericycle averaging five cells in thickness, in which are irregular groups of stone cells. Around the pericycle is the primary cortex, averaging seven or eight cells in depth. These have suberized cell walls, and appear to be in the process of direct transformation into cork; no definite cork cambium is to be observed. However, the inner of the suberized cells were still alive, having in them functioning nuclei. (See fig. 55.)

The epidermis is a single layer of cells. (See figs. 53 and 54.) One peculiarity of the root hairs is that they retain their place long after the inner tissues have become suberized and they are functioning no longer. (Figs. 47, 53, and 54.)

The xylem is made up of tracheal tubes, tracheids, wood fibers, and sclerotic wood parenchyma cells. (See figs. 49, 50, and 51.) The tracheal elements consist largely of tracheal tubes having many small bordered pits, and occasional ridges in the walls. Here, as in the stem xylem, were noted at more or less regular intervals the thickened rings in the tubes. (See fig. 51.) The other xylem tissues and those of the phloëm do not differ essentially from those of the stem, so they will not need to be discussed further here. The root hairs are numerous, and as stated above, are remarkable for the tenacity of their hold, even after the transformation of the cortical tissue into cork has cut them off from nourishment and caused them to die. They are found, after this condition has come about, to be suberized on the outer portion of their walls with a cellulose inner layer.

RESUME.

The plant, as a whole, is xerophytic. Its characters may be summarized as follows:

PROTECTIVE DEVICES:
- Low habit of growth.
- Small leaves.
- Heavy hairy covering.
- Heavy cutinization of all exposed surfaces.
- Secreting glands and spaces.

STRENGTHENING TISSUES:
- Anastomosing bast ring.
- High lignification.
- Complete silicification.
- Heavy secondary thickening of pith region.

WATER PROVISIONS:
- Thick walled and plain pitted storage cells in leaf.
- Restricted evaporating surface.
- Extensive tracheal storage system.

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THE

KANSAS UNIVERSITY SCIENCE BULLETIN.

Vol. VII, No. 17—June, 1913.

(Whole Series, Vol. XVII, No. 17.)

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PUBLISHED BY THE UNIVERSITY,

LAWRENCE, KAN.

Entered at the post-office in Lawrence as second-class matter

4-5259

THE KANSAS UNIVERSITY SCIENCE BULLETIN.

VOL. VII, No. 17] JUNE, 1913. [WHOLE SERIES VOL. XVII, No 17

STUDIES ON THE ANATOMY OF ALFALFA.

(*Medicago sativa* L.)

BY ORVILLE T. WILSON.

Plates LV LIX

INTRODUCTION.

DURING the summer session of 1911 and the college semesters 1911-'12, the writer was engaged in graduate work at the University of Kansas. Considerable time was spent during that period in working out the structures of alfalfa, under the personal direction of Prof. W. C. Stevens. The results of these studies are embodied in the following pages.

LABORATORY METHODS EMPLOYED.

The methods employed in the studies were mainly those described in Stevens's Plant Anatomy. The bleaching and staining of the leaf were done in accordance with directions by Mr. L. M. Peace in the *Plant World* of April, 1910. Free-hand sections of material, both fresh and preserved in formalin, paraffine sections in various stains, and macerated sections were used in studying the tissues. The drawings were most frequently made with a camera lucida attachment, but in some cases a projection lantern was used in drawing. When it seemed desirable for the sake of clearness, drawings were somewhat conventionalized.

THE ROOT.

The root of alfalfa is usually a tap root, the main axis penetrating to a depth of six to twelve feet, sometimes twenty or more feet. In some cases the tap root branches a short distance

Received for publication May 29, 1913.

below the surface, giving rise to a group of heavy, downward-directed secondary roots. Such a root system affords an extensive connection with the soil reservoir, and partly accounts for the ability of a stand to produce three to five crops in a summer. In this extensive root system also, a considerable amount of food storage over winter is made possible. The upper portions of the roots were found to be packed with food in the fall. In the spring this food was used in the new growth. During the winter a pronounced concentration of food at the bases of the buds was noticeable. Although in no way exceptional structurally, the anatomy of the root may be briefly considered.

A cross section of the young root (fig. 1) shows a protostele of the primitive exarch type. A protoxylem point is shown at (*c*). In the later stage a circle of collateral vascular bundles is apparent, with pith rays radiating from the original protoxylem points. At this stage the root is much like the stem in general appearance of the cross section. Just outside the phloëm is a cylinder of pericycle surrounded by larger cells (*d*) with more or less lignified walls. This cylinder lends strength to the root and protection to the tender phloëm region. Such a protective tissue is especially serviceable, since the outer or cortical region (*c*) of the bark soon breaks down as shown.

THE STEM.

Alfalfa is a perennial herb, giving rise each spring to a crown of stems from a persistent root system. This crown arrangement of stems accounts for the luxuriant cover of growth in an alfalfa field, when the single plants may be a foot or more apart.

A cross section of a stem of alfalfa appears as shown in figure 2. The epidermis is multiple, being made up of two layers, quite similar in cell structure. (Fig. 2, *f*.) The outer layer of cells has a more thickened and cutinized outer wall. The corners of the somewhat four-angled stem show a multiplication of the cell layers under the epidermis, seemingly serving for greater strength. (Fig. 2, *g*.) Between the epidermal region and the groups of bast fibers (fig. 2, *h*) is an extensive tissue (fig. 2, *i*), which bears numerous chloroplasts in the cells. The chloroplasts are not shown in the figure. The groups of bast fibers (fig. 2, *h*) are just outside the phloëm region

(fig. 2, *j*) of the vascular bundles; the xylem, as usual, is inside the phloëm region. Inside the vascular bundles is the pith (fig. 2, *l*).

The cells of the stem epidermis are, in the main, more or less irregular, and elongated somewhat in the direction of the stem axis (fig. 3, *g*). Among these cells occur numerous stomata (fig. 3, *h*). Alfalfa is especially well provided with these, the number on the stem being about 300 to the sq. mm. The walls of the guard cells do not show as much cutinization as do those of the ordinary epidermal cells.

Numerous trichomes also arise from the stem epidermis. Some of these are two-celled glandular structures, with no apparent content (fig. 3, *f*); others seem to contain active protoplasts (fig. 4, *m*). A third class of hairs, more numerous, are similar to those found upon the leaves (fig. 5). The glandular hairs occur about 24 to a sq. mm.; and the elongated hairs are somewhat more numerous. No vital connection has been found between the trichomes and the activities of the plant, although their presence or absence may serve in distinguishing varieties. The glandular type of hairs is not widely distributed among the other genera of the Papilionaceæ. Of the forms mentioned by Solereder, those of the alfalfa would seem to be "small, club-shaped external glands, composed of few cells and without a distinctly marked stalk."

Underneath the outer epidermal cells is a layer of cells much like them, except that the outer walls are not cutinized or thickened to so great an extent (fig. 7, *n*). The corners of the stem are strengthened by a tissue (fig. 2, *g*), extending from the epidermis to the lignified tissue surrounding the corner vascular bundle. This arrangement makes the angles of the stem especially strong and leaves four regions of chlorophyll tissue (fig. 7, *o*) between the bundles of the bast and the epidermis on each side of the stem. In these chlorophyll regions there are numerous air spaces between the cells. There are several layers of the chloroplast-bearing cells. The chloroplasts are of the usual lens-shaped type, peripherally arranged. In this region, rich in protein, the main food value of the alfalfa stem lies. Solereder, in discussing the presence of such chlorophyll regions in the stems of Papilionaceæ, says: "In many woody species, principally those with reduced leaves, much of the primary cortex is developed as palisade parenchyma or assimilatory tissue of some other form. Associated

with this feature we find (*a*) abundant stomata and (*b*) strands of sclerenchyma fibers either in the cortex or connected with the fibro-vascular system." While the stem of alfalfa would be classed as herbaceous rather than woody, the above description will be seen to apply to it in most respects.

The chlorophyll region is bounded on the inside by the groups of bast fibers (fig. 2, *h*), and by lignified sclerenchyma cells, or thin-walled parenchyma (fig. 6 *c*), which border the groups of bast fibers on the sides and occupy the region between the vascular bundles, forming the medullary rays.

The vascular bundles may now be considered more in detail. At the exterior of each bundle in the pericycle is a layer of cells with thickened walls, these walls being lignified, at least on the side towards the bundle, and classed as stone cells because of the unusual hardening and thickening of the walls (fig. 9, *e*). They are considered here because of their intimate association with the vascular bundles. In these stone cells may usually be seen embedded crystals of calcium oxalate. (Fig. 9, *i*.) These crystals are simple in their occurrence, clustered crystals being absent in the Papilionaceæ. Enlarged crystals are shown in fig. 10. This sheath of stone cells, connected as it is with those about adjoining bundles by cells with lignified walls, is part of an almost continuous sclerenchyma sheath surrounding the whole system of vascular bundles. Such sclerenchymatous sheaths are reported by Solereder as occurring in a number of the Papilionaceæ.

Immediately inside the crystal-containing sheath just mentioned may be found, just outside the phloëm of each bundle, a tissue of compactly associated thick-walled cells, the bast fibers (fig. 2, *h*). They often are arranged in the Papilionaceæ as here found. The bast fibers are greatly elongated and overlap at the ends (fig. 11, *j* and *k*), forming a firm, continuous tissue throughout the length of the stem and extending into the main veins of the leaf. The groups are three or four cells in thickness, as a rule. At the corners they are more numerous (fig. 6, *a*) than on the sides. They are marked by numerous simple pits (fig. 12, *m*). The walls of the fibers are lignified, but not so completely as is the xylem woody tissue. The presence of the simple pits in the fibers is characteristic of the Papilionaceæ. Solereder, quoting from Jannicke on Herbaceous Stems, says that in most members of the genus *Trifolium*, a near relative of *Medicago*, the vascular bundles are united by inter-

fascicular tissue composed of parenchymatous sclerenchyma, which is connected externally with the groups of hard bast belonging to the vascular bundles. He also mentions that in some species of *Medicago* the interfascicular tissue consists of sclerenchymatous prosenchyma and parenchyma. In alfalfa we have the prosenchyma distributed into the groups of fibers discussed above, and these groups, as well as the bundles as a whole, connected by sclerenchymatous parenchyma. (Fig. 2.)

The essential elements of the vascular bundles, the water tubes and the soft bast, are arranged collaterally. The phloëm (fig. 9, *o*, and fig. 7, *p*) is made up of sieve tubes (fig. 18) and phloëm parenchyma, together constituting the soft bast. This region is rich in nutritive value. The sieve tubes are rows of very narrow, elongated cells (fig. 18), with end-wall sieve plates (fig. 18, *b*) of the usual type. The tubes are bound together by a network of thin-walled phloëm parenchyma cells. According to Moeller, short articulations and the exclusive occurrence of simple sieve plates are characteristic of the sieve tubes.

The xylem of alfalfa forms an almost continuous cylinder around the mature stem. Between the bundles the xylem portions are connected by lignified parenchyma cells (fig. 7, *r*). The water tubes and tracheids are generally about three rows deep in the bundles. The vessels are frequently bound together by radial rows of wood parenchyma cells (fig. 7, *y*) or by wood fibers. The spiral thickening has a marked border in some of the tubes.

About half as numerous as the spirally thickened tubes (fig. 9, *x*) are the rows of pitted tracheids (fig. 9, *w*), which are found next to the phloëm. The pits are typically bordered (fig. 9, *w*). These occur sometimes in straight rows, sometimes in spiral arrangement. They are most numerous in the vessel wall where it is in contact with ray or wood parenchyma. When arranged in straight rows there are usually three rows down the tracheid, the number of pits to a mm. in length being about 900. In some of the pits peculiarities of structure were observed (fig. 13, *t*). These may be the same as the sieve-pit-like structures of bordered pits found by Johnson in *Medicago*, *Lupinus*, and *Melilotus*, and mentioned by Solereder.

The bundles at the corners show a diminution of both sieve tubes and water tubes, the associated bast and wood fibers being especially developed. (See fig. 6.)

The pith (fig. 1, *l*) is very prominent. The large cells in the central portion with unlignified walls (fig. 2, *w*) are in contrast to the smaller cells near the bundles with walls lignified. A peculiar arrangement is to be noticed at the corners (fig. 6). Just inside the xylem is a region of unlignified cells (fig. 6, *v*), resembling an inner phloëm, and surrounded, toward the pith, by a region of lignified pith cells (fig. 6, *p*). The pith contains numerous air spaces (fig. 2, *g*), but no extensive cavities of any kind.

THE LEAF.

The leaves of alfalfa are pinnately compound, each petiole bearing three leaflets, the characteristic leaf-type of clover. Each leaflet has a small petiolule, is elliptical in general outline, much more narrow than long, and with prominent teeth on the margins towards the tips (fig. 41, *s*). It is very noticeable that these teeth are the terminating points for the main branches of the system of veins (fig. 41). These terminations may be of the nature of hydatodes, as they seemed in some plants to act as drip tips. Beads of water were observed to accumulate at these tips after the plant was covered with a bell jar over night.

At the base of the petiole are attached two stipules, which are leaf-like in structure, and probably do some of the ordinary work of leaves. In cross section (fig. 24) the petiole appears heart-shaped, and the three vascular bundles (fig. 24, *z*), which extend into the three leaflets to form their midribs, are arranged in an arc. This arrangement is quite characteristic in the Papilionaceæ with three leaflets. No medullary rays are found in the petioles of any of the Papilionaceæ. Each leaflet petiole (fig. 25) is much like the leaf petiole, except that in section only one vascular bundle appears (fig. 35). The cells next the epidermis are full of stored food material, mostly starch.

The epidermis of the leaf is made up of irregularly shaped cells (fig. 14), the outer walls of which are strongly cutinized (fig. 17). There are numerous trichomes on both surfaces of the leaf, as well as stomata, both being slightly more numerous on the under surface.

The trichomes of the leaf are two-celled structures (fig. 15), the basal cell being in the same plane with the epidermis, and the outer cell greatly elongated. These trichomes are similar to the most numerous type on the stem. The elongated portion is marked on the margin by numerous papillæ-like swellings (fig. 15, *c*). Solereder says concerning such markings: "According to Vogelsberger, in certain species of *Ebenus* the end cell of the trichome under discussion is provided with small, tooth-like papillæ, which are directed towards the apex of the hair." Similar structures were observed by Schenck in *Medicago arborea*. In alfalfa the projections are directed at right angles to the surface rather than towards the apex. It is of interest to note here that the variety of alfalfa most resistant to drought, the Arabian, is the most hairy economic variety. The average length of alfalfa hairs is about .7 mm., the diameter about .01 mm., the number of hairs to a sq. mm. being about 24. The ordinary observer would hardly consider the alfalfa leaf a hairy one.

Exceptions to the general irregularity of epidermal cell arrangement are found under the midrib and at the bases of the trichomes. Under the midrib the cells are elongated parallel with it; they are narrow and strongly cutinized, as well as roughened on the outer wall. No stomata are found here, the general arrangement apparently serving to strengthen the midrib of the leaf, and also being in harmony with its conductive function. At the bases of the trichomes the epidermal cells form rosette patterns (fig. 15), with the basal cell of the trichomes as a center.

The stomata of the leaf are like those of the stem, but more numerous, in places as many as 700 to a sq. mm. being found. The exceptionally high number is to some extent an indication of the leaf's exceptional functional activity.

Under the stomata are extensive air spaces (fig. 22, *e* and *f*). The palisade layer (fig. 22, *g*) is also well aërated by intercellular spaces.

Under the palisade tissue, and extending to the lower epidermis, is the spongy parenchyma (fig. 22, *h*). The loose arrangement of thin-walled cells containing chloroplasts is typical of this region in leaves. Air spaces in this region communicate freely with the numerous stomata on the under side of the leaf.

Embedded in the spongy mesophyll is the venation of the leaf (fig. 41). Each leaflet is provided with one prominent midrib (fig. 41, *z*), which gives rise to lateral branches extending to the margin, these lateral branches giving rise to smaller branches, and so on, the whole system forming a close network of veins. The veinlets lie exceptionally close to the lower epidermis (fig. 22, *a*), the surrounding parenchyma or sclerenchyma sheath often being in contact with the lower epidermal cells. The surrounding sheath (fig. 38) is similar in many respects to that described as surrounding the bundles of the stem, and corresponds to it. Lignification is not so extensive in the sheath cells of the leaf, but the crystals are found in numbers in the thick-walled cells of the parenchyma sheath cells. The sheath about the midrib shows marked sclerosis (fig. 23), as do also the sheaths about the main lateral veins extending from the midrib. Toward the tips of the latter the specially modified cells give way to ordinary border parenchyma cells (figs. 27 and 28), such as border the finer veinlets, and are common to the veining systems of many leaves. The tips of the veinlets which end freely in the mesophyll (figs. 29 and 30), unlike those of the midrib and lateral veins ending in the margin of the leaf, are made up of various compact groups of spirally thickened tracheids (fig. 31).

TABULATED CELL DIMENSIONS IN MM.

	Cross section	Longitudinal section
ROOT.		
Endodermis	.01 x .017	
Phloëm parenchyma	.0052 x .005	
Water tubes	.012 x .012	
STEM.		
Epidermis, outer	.012 x .008	
Epidermis, inner	.01 x .007	
Stomata, guard cells	.02 x .0075	
Hairs, elongated, basal cells	.021 x .021	
Hairs, elongated, tip cells	.014 x .014	.014 x .7
Hairs, glandular, neck cell	.008 x .008	.008 x .04
Hairs, glandular, tip cell	.016 x .016	.016 x .04
Chlorophyll cells	.009 x .009	.009 x .01
Strengthening cells at corners	.008 x .004	
Sclerenchyma sheath cells	.017 x .02	
Pith cells	.04 x .04	.04 x .05
Sclerenchymatous parenchyma	.015 x .015	.01 x .015
Xylem tubes	.017 x .017	
Xylem parenchyma	.005 x .005	
Sieve tubes	.004 x .004	.004 x .032
Bast fibers	.012 x .01	.012 x .25
LEAF.		
Epidermis	.03 x .015	
Palisade cells	.015 x .015	.04 x .015
Tracheids	.015 x .015	.015 x .08
Spongy parenchyma	.021 x .017	.021 x .02
Border parenchyma	.03 x .03	.03 x .022

PLATE I.

The geographical distribution of *Enchodus* is indicated by the dots on the map.

PLATE I.

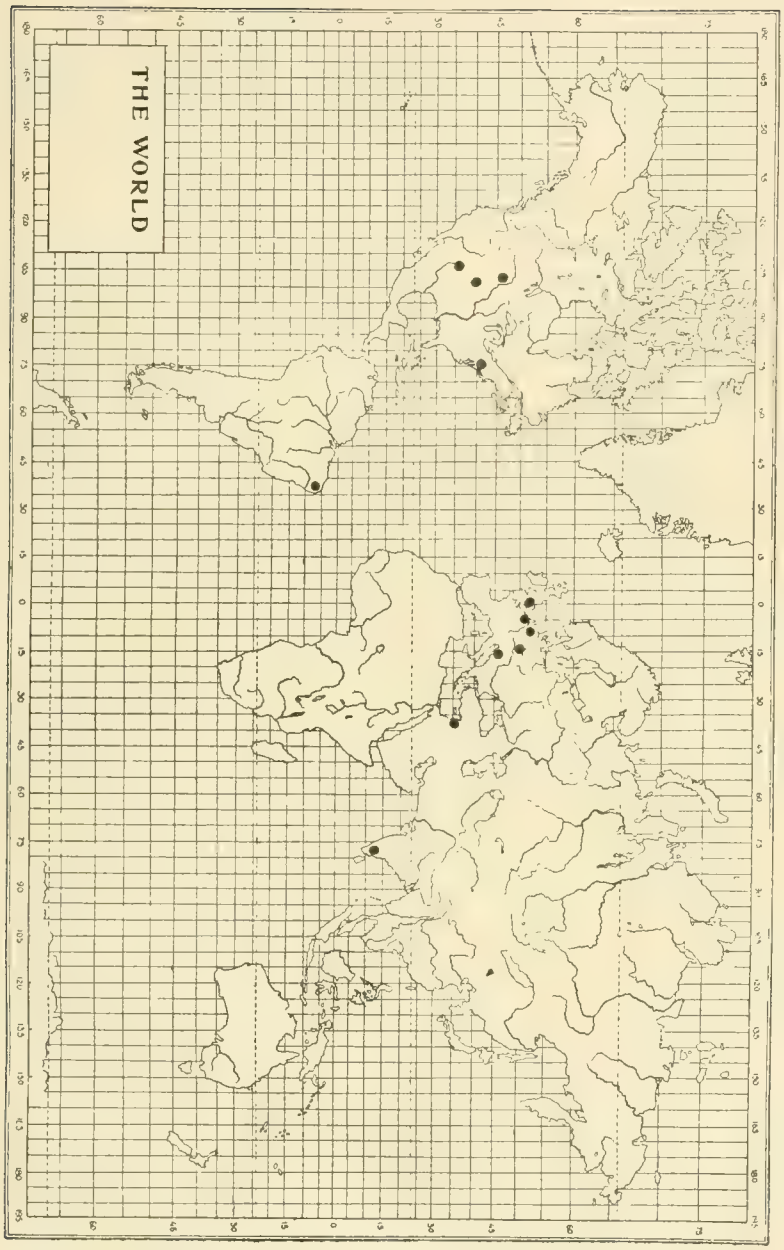

PLATE II.

Enchodus petrosus Cope.

FIGURE 1.—*boc*, basioccipital; *eoc*, exoccipital; *e*, diagram of occipital condyle, trisected by sutures, the upper two-thirds being the exoccipitals, and the lower third the basioccipital; *opi*, opisthotic; *op*, opercular; *pa*, parasphenoid; *pal*, left palatine; *pal*, ventral view; *pald*, dorsal view of same (a mesial view of this palatine is shown in plate III, fig. 1); *pro*, proötic; *pr*, pterotic; *sp*, sphenotic. × ⅓. No. 804.

FIG. 2.—Portion of right dentary figured in plate III, figure 9, showing the contour, *f-f*, or the alveolar border in a vertical plane passing through the middle of the scars of the discarded teeth. *a* is the foramen of the blood vessel supplying the fang. The broken lines at the base of the fang indicate the position of the former bases of fangs, in a median section through this portion of the dentary, *n* being the base of the last discarded fang, and *o* that of the next to the last. *b*, *c*, *d* and *e* are the bases of the posterior teeth. There is to be seen one scar in front of *b*, two between *b* and *c*, and three between *d* and *e*. *d* leans cephalad over the scar at its base, and this scar appears to be crowded forward. The dotted portion in the lower part of the figure is plaster of Paris. × 1/1. No. 803.

FIGS. 3, 4, AND 5.—An opercular, supraclavicle, and portions of a dentary and articular of a species much larger than any other in the collection. × ⅓. No. 805.

PLATE II.

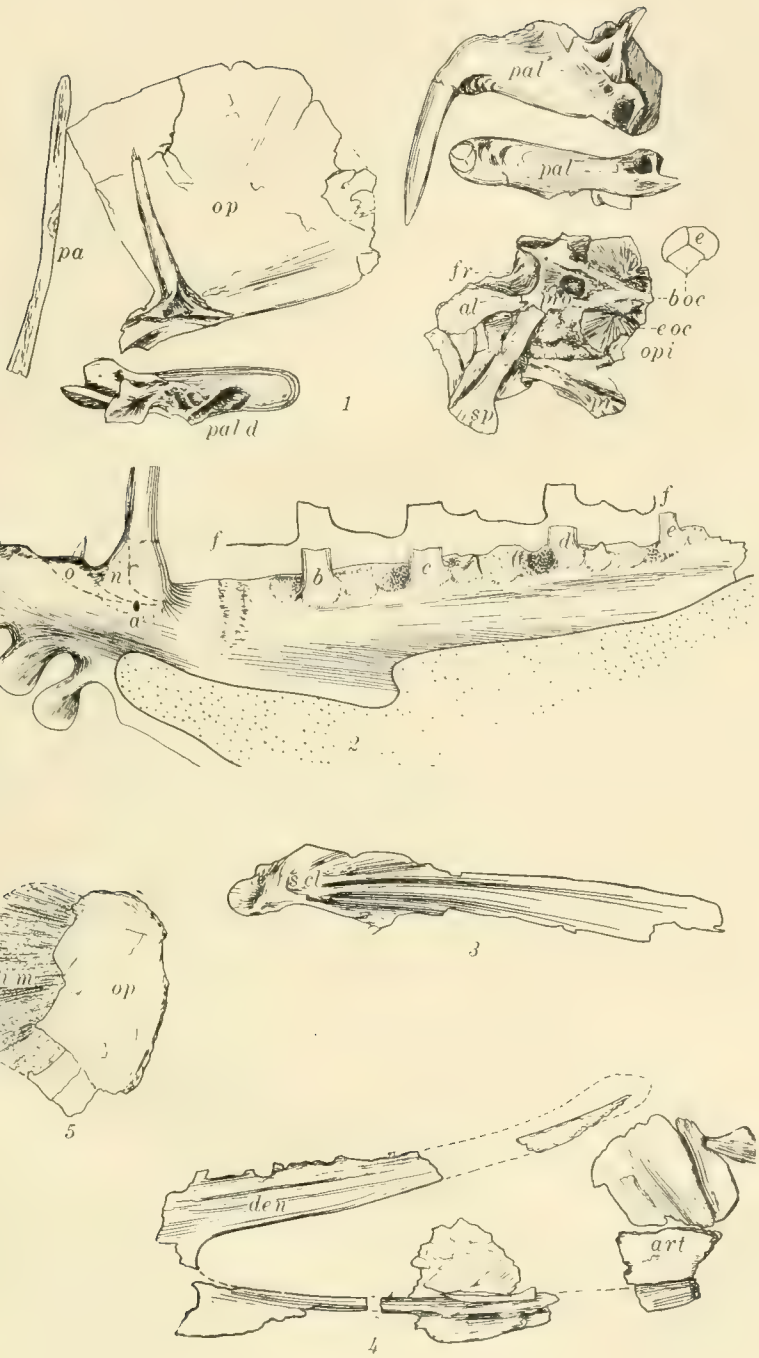
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PLATE III.

Enchodus petrosus Cope.

FIGURE 1.—Two palatines; upper figure is left bone. This belongs to specimen No. 804 (pl. II). *eth*, part of ethmoid in position; *pt*, anterior end of pterygoid in position. × ⅓. No. 80.

FIG. 2.—Pair palatines arranged as in figure 1. × ⅓. No. 803.

FIG. 3.—Separate palatine tooth with a cross section of the same at the middle point just above it. The tooth shows the extent of the grooves on the posterior mesial side. × ⅓. No. 833.

FIG. 4.—Pair of palatines without fangs, arranged as in figure 1. × ⅓. No. 801.

FIG. 5.—Two odd palatines arranged as in figure 1. The upper figure may not be *E. petrosus*. (cf. pl. VII, fig. 17.) × ⅓. No. 16.

FIG. 6.—Premaxillary. × ⅓. No. 803.

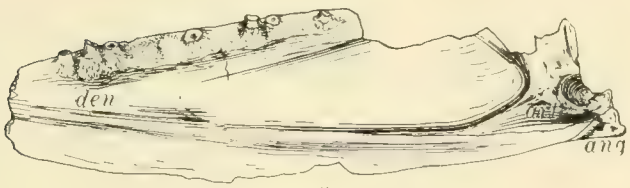
FIG. 7.—Symphysial portion of left dentary. × ⅓. No. 803.

FIG. 8.—Mandible, showing mesial aspect. *art*, articular; *ang*, angular; *den*, dentary. × ⅓. No. 839.

FIG. 9.—Elements of the mandibles. The quadrate (*qu*) is raised out of its position in order to show it more clearly. *art*, articular; *ang*, angular; *den*, dentary; *pal*, palatine; *pt*, pterygoid. × ⅓. No. 803.

FIGS. 10 AND 11.—The largest and smallest of a graduated series of palatines which can not be positively identified, as the fangs are broken off. They are probably *E. petrosus*. × ⅓. Nos. 21 and 25.

PLATE III.



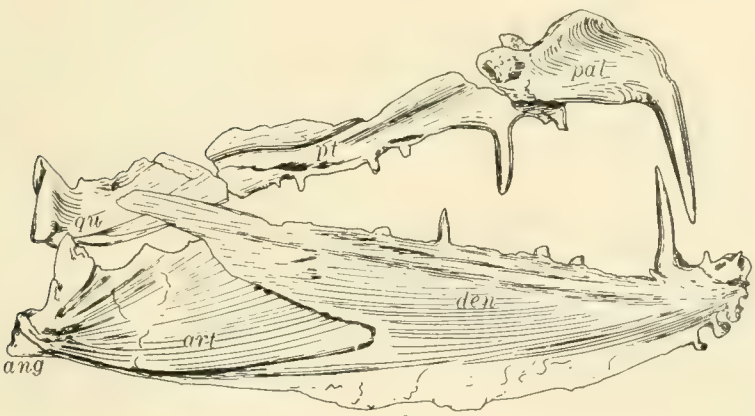


PLATE IV.

Enchodus petrosus Cope.

portion of the dentary; *abv*, abdominal vertebræ; *cl*, clavicle; *cdr*, caudal fin portion of the dentary; *abv*, abdominal vertebræ; *cl*, clavicle; *cdr*, caudal fin rays; *ch*, ceratohyal; *den'*, left dentary; *den*, right dentary; *eth*, ethmoid; *fr*, right frontal; *fr'*, portion of left frontal; *hc*, hypercoracoid; *hco*, hypocoracoid; *op*, opercular; *pal*, right palatine; *pal'*, left palatine; *pt*, pterygoid; *pst*, post-temporal; *pr*, *pr*, pterotics; *pmx*, premaxillary; *pt*, cephalic end of pterygoid attached to left palatine; *pc*, pectoral rays; *pa*, parasphenoid; *cs*, cross section of the same at the point indicated by the broken line; *pop*, preopercular; *qu*, quadrate; *qu'* right quadrate; *r*, ribs; *scl*, dorsal end of supraclavicle; *sop*, subopercular; *sp*, *sp*, sphenotics; *vt*, vertebræ; *x*, probably the caudal end of the parasphenoid. × ⅓. No. 836.

PLATE IV.

pa
cs
den
s.op
op
cl
r
qu
ar
pop
pt
pa
h.co
cd r
ch
den
pst
st
pr
pal
eth
pmx
fr
vt
sq

PLATE V.

Enchodus petrosus Cope.

Showing fragmentary skull and complete spinal column: *abv*, abdominal vertebræ; *art*, left articular; *ang*, angular; *br*, branchiostegals; *cdr*, caudal rays; *cdv*, caudal vertebræ; *den*, dentary; *fr*, frontal; *hm*, hyomandibular; *h*, hæmapophysis; *hs*, hæmal spines; *hy*, hypural; *iop*, interopercular; *ms*, mesopterygoid; *mt*, left metapterygoid; *mt'*, right metapterygoid; *mx*, maxillary; *ns*, neural spine; *n*, neurapophysis; *op*, opercular; *pal*, palatine; *ph*, a pharyngeal; *ph'*, probably one of the pharyngeals; *pmx*, premaxillary; *pop*, preopercular; *pst*, post-temporal; *por*, preorbital; *r*, ribs; *sop*, subopercular; *x*, undetermined bone; *y*, probably a portion of the post-temporal; *scls*, supraclavicles. × ¼. No. 807.

PLATE V.

PLATE VI.

Enchodus petrosus Cope.

FIGURE 1.—Almost complete tail: *cdv*, caudal vertebra; *cdr*, caudal fin rays; *hy*, hypural; *h*, hæmapophysis; *n*, neurapophysis. × ½. No. 806.

FIG. 2.—Right palatine fang, showing the scars left by former fangs. × ½. No. 126.

Enchodus dirus Leidy.

FIG. 3.—Portion of cranium, right view: *ang*, angular; *art*, articular; *art'*, left articular; *br*, branchiostegals; *den*, dentary; *eth*, ethmoid; *f*, fang of palatine which lies under the premaxillary; *hc*, hypercoracoid; *hm*, hyomandibular; *ms*, mesopterygoid; *mt*, metapterygoid; *mx*, maxillary; *op*, opercular; *pop*, preopercular; *qu*, quadrate; *sop*, subopercular. × ½. No. 817.

FIG. 4.—Same as figure 3, left view: *pal*, left palatine; *pal'*, right palatine; *c*, ceratohyal; *pt*, pterygoid; *g*, girdle bone, probably the clavicle. × ½. No. 817.

PLATE VI.

cd v
h
hy
1
2
hm
eth
pmx
mx
ms
mt
pmx
op
sop
den
pop
art
hc
art
ang
br
3
pmx
hm
eth
pal
pal
pop
hc
g
den
art
br

PLATE VII.

Enchodus petrosus Cope.

FIGURE 1.—Diagram of the palatine bone, lateral view, showing the course of the larger blood vessels, represented in black, supplying the fang and the caudal end of the bone. The lines *A-0*, *A-1*, *A-2*, etc., represent planes of horizontal sections shown in the following figures, which are numbered to correspond. The sections were obtained by grinding off the bone slowly, beginning on the ventral side, and, at the various levels indicated, carefully drawing the surface exposed. After reaching the level *A-9*, the bone was ground parallel to this plane until the level indicated by line *B-X* was reached, after which *B* remains constant, the planes of succeeding sections, *10* to *14*, all passing through it. *a*, *b*, *c*, *d*, *e*, *f*, *g* and *h* represent the ventral ends of a number of canals in the osteodentine, which served successively as the main blood supply of the great palatine tooth. Before the tooth was sectioned the ventral ends of these canals were indicated by dark spots on the surface, one occurring about the center of each crescentic scar. (See fig. 15, pl. XV.) These canals are found to proceed dorso-caudad, and soon the posterior five anastomose and change their course to dorso-mesio-cephalad and emerge on the mesial surface of the bone at the point *i*. The canals beginning at *b* and *c* proceed in the same general direction as those mentioned, and, after anastomosing, open to the mesial surface at two places very near point *B* (also indicated in fig. 8, *cb*); *a* marks the open end of the canal which led to the fang, the latter having been broken off of this palatine. This canal was traced to a point at the level of *A-8*, where it opened to the mesial surface (fig. 9, *a*). The outline of the basal portion of the fang is shown, and the continuation of the canal down the fang is indicated by a broken line. The large vessels in the caudal end of the bone are shown to belong to a system distinct from the canals in the osteodentine. Its main trunk proceeds dorsally through the osteodentine, giving off few branches until it reaches the true bond above, when it is found to branch profusely. At about the level *A-9* its cross section was found to elongate cephalo-caudally. At the level *B-10*, it has divided into four or five large vessels having an elongate cross section and many small anastomosing vessels. Continuing dorsad, the vessels become slightly grouped at the level *B-12*, and at the level *B-14* the five larger ones are in cross section as shown in the section taken at this place (fig. 14), and, retaining the same size, shape and position relative to each other, they now proceed dorso-cephalad until they emerge on the dorsal surface of the palatine. The outer edges of four of these parallel canals are represented in the diagram (fig. 1); the fifth one (*g*, fig. 14), being situated directly mesial to the others, could not be shown in the diagram. Only the edges are represented, as the canals are elongated in cross section and overlap so that they can not be shown full size (figs. 13, 14). Most of the palatines show the openings of these canals on the dorsal surface.

FIGURES 2 TO 14, inclusive, are numbered to correspond to the various levels indicated in the diagram, indicating horizontal sections taken at these levels. The external side of the bone is uppermost in the sections. Cross sections of canals are represented in black. Point *A*, figure 1, corresponds to line *A-A* in figures 2 to 9, inclusive; point *B* in figure 1 to line *B-B* in figures 10 to 14, inclusive. The lettering in all of

the sections corresponds to that in the diagram, figure 1, so that *a*, section 2, is the cross sections of canal *a*, figure 1, etc., *hgf* being a cross section of the united vessels *h*, *g*, and *f*, figure 1. *V* of section 2 is the cross section of the large canal in the caudal portion of the palatine. Section 3 differs only slightly from 2. Here *e* has united with *hgf* to form *hgfe*, and *d* is about to unite. Section 4 shows *d* united, and the system opening out to the surface at *i*. (See point *i*, fig. 1.) The areas bounded by curved lines in all of the sections, and marked *a*, *b*, *c*, *d*, etc., in section 4, represent cross sections of bases of the successive fangs. Note the shape of these areas near the caudal end of the sections, showing that more than half of the external portion has been absorbed. In section 2, very near the upper margin of the figure, are shown five tiny openings on the external side of the bone, one corresponding to each of the areas *b*, *c*, *d*, *e*, and *f*. These appear on the margins of the sections in figures 3 and 4, well on the ground surface in figures *5* and *6*, and are not to be seen in section 7 or thereafter. Not many of the palatines show the openings of this series of canals. Section 5 shows *h*, *g*, *f*, *e* and *d* disconnected, and canal *a* very near the mesial surface. Figure 6 shows no important change. Figure 7 shows canal *a* divided. Figure 8 shows canals *c* and *b* united to form *cb*, which opens to the mesial surface. The section shown in figure 9 passes above canals *c* and *b*, and shows *a* about to open to the surface. *V* has become much elongated in cross section. No drawing was made at level *B-X*. In section 10, *V* has branched to form the group *M*, and *a* has opened to the mesial surface, as shown in all sections following, the portion cephalad to the line *B-B* not being ground any further, as shown in the diagram, figure 1. Section 11 indicates further branching of the canals in the caudal end of the bone. Section 12 shows these canals slightly grouped again, and in figure 13 the four external canals show much elongated cross sections. Figure 14 shows these four canals slightly enlarged, and a fifth one (*g*) just mesial to them. These five proceed dorso-cephalad to the surface, as shown in figure 1. The oval cavity (*h*) is not a canal, but only a space in the bone, probably abnormal.

FIGS. 15 AND 16.—Figure 15 shows a longitudinal vertical section of a palatine, not quite median, but taken along the line *D-D* in figure 16, which is a ventral view of the same bone after the portion ventral to the line *C-C* (fig. 15) had been ground away. The areas *a*, *b*, *c*, *d*, *e*, *f*, etc., represent the vertical sections of the successive bases of the palatine fangs, thus showing the relation of the osteodentine of which they are composed to the true bone above. The lettering in figure 16 corresponds to that of figure 1. × ½. No. 837.

FIG. 17.—A longitudinal vertical section of a palatine which has been described and figured as *E. petrosus* by Stewart (1900). A mesial view of the same bone is shown in plate III, figure 5, upper figure. This palatine differs from figure 15 in having a dorso-cephalic process, and in the different outlines of the vertical sections of the successive osteodentine bases. The dorsal ends of the first five, *a*, *b*, *c*, *d*, *e*, are bent forward. Bases *b* and *e* seem to have had some irregularity in growth. Evidently the process above base *a* is not simply a stage in the production of the fang, but is a character that was always present in this specimen and never occurred in the development of the palatine shown in figure 15. × ⅔. No. 16.

PLATE VII.

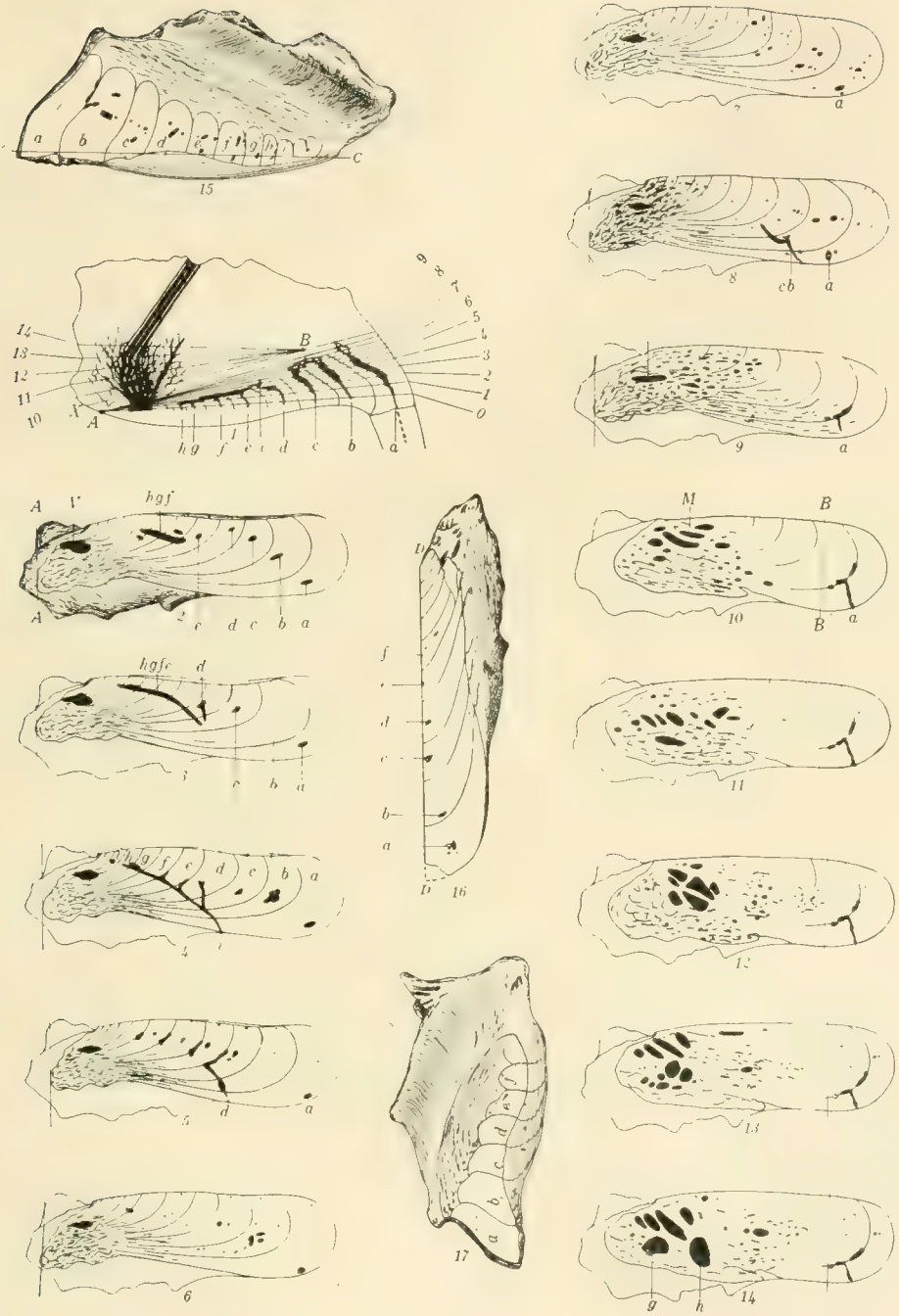


PLATE VIII.

Enchodus petrosus Cope.

This specimen, No. 198 in the American Museum, has been figured in part, and described as a new species (*E. Sævus*) by Dr. O. P. Hay, but since it displays several features heretofore not discussed, it was thought best to refigure it.

vt, a vertebra. The fragmentary hyomandibulars (*hm*) are placed with their dorsal ends upward and their cephalic edges inward on the plate. The larger bone between them represents the dorsal aspect of the caudal end of the skull; *fr*, the left frontal. The sutures in the skull are too indistinct to be traced. The broken lines represent approximately the outer margins of the cephalic ends of the frontals. These bones should join firmly to the ethmoid, *eth* (cf. pl. XI, fig. 1, and pl. VI, figs. 3 and 4). *pal*, palatine; *pmx*, the premaxillaries. The cephalic ends are placed downward on the plate. In life the mesial edges probably approximated each other. Near the mesial edges are two elongated foramina, the right one nearly one-fourth longer than the left. On the cephalo-ventral edges are the bases of a single row of teeth.

The left palatine of this specimen furnishes a fine illustration of the manner in which the new fang originates. The old fang has been broken off, but a considerable portion of the new base remains. This is much flattened on the cephalic side, obviously due to the absorption of the osteodentine. Over this flattened surface there has been deposited a porous layer of new bone tissue, which in time evidently would have hardened and thickened sufficiently to form the base for a new fang. This is the only specimen examined by the author, which shows this particular state in the replacement of the palatine fang. It is doubtless perfectly normal and furnishes a substantial corroboration of the theory of the replacement of palatine fangs in *Enchodus*.

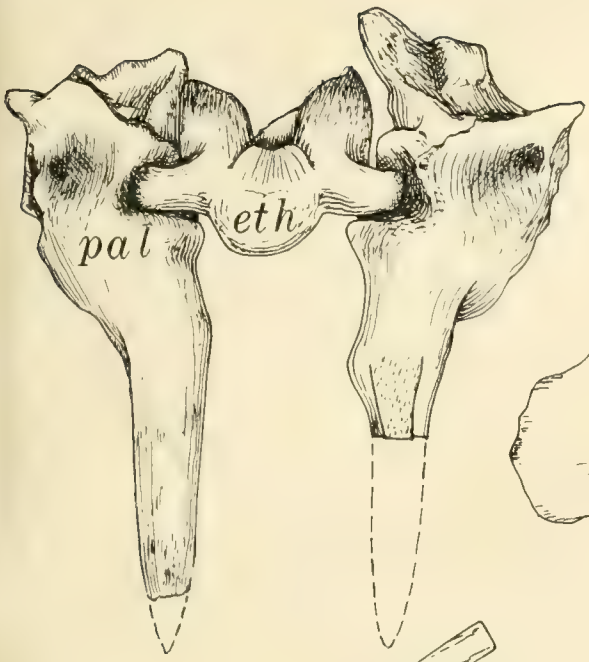
PLATE VIII.

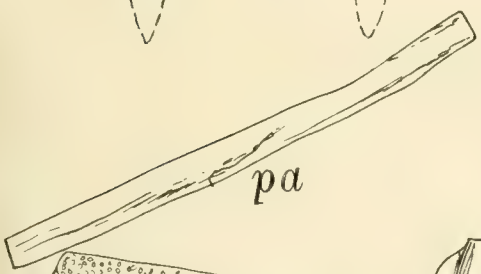
PLATE IX.

Enchodus petrosus Cope.

The pair of palatines represented on this plate is the same as that shown on plate VIII, viz., specimen No. 198 of the American Museum. In this cephalic view the layer of new bone tissue on the end of the left palatine is shown a little more distinctly than in plate VIII. In the process of removing the gypsum crystals from this portion of the bone, the lateral margins of the layer were broken off before the nature of it was discovered. The upper opercular (*op*) belongs to the same specimen. The lower of the opercular bones (*op*) belongs to specimen No. 804 in the University of Kansas Museum. The upper portion of this opercular is absent, but the similarity of the tuberculation and ridges on the outer surface of these two operculars is so striking that there seems little doubt that they belong to the same species. While on the opercular of the American Museum specimen there is only a small area just caudad to the point of suspension to the hyomandibular, which shows the tubercles distinctly, this is to be accounted for by the fact that most of the surface of this opercular was injured by the crystallized gypsum with which the bones of this specimen were originally enveloped. The larger ridges extend mostly dorsally and ventrally, gradually becoming smaller and more tuberculated caudally, the intermediate area being much tuberculated, but free from ridges except near the point of suspension, where they are very short and small. *mt*, metapterygoid; *vt*, vertebra.

PLATE IX.





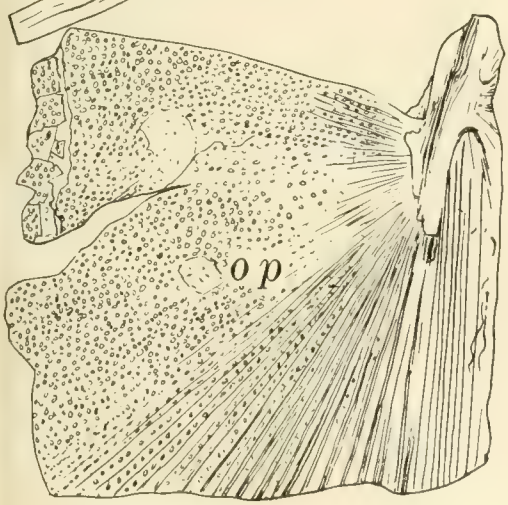


PLATE X.

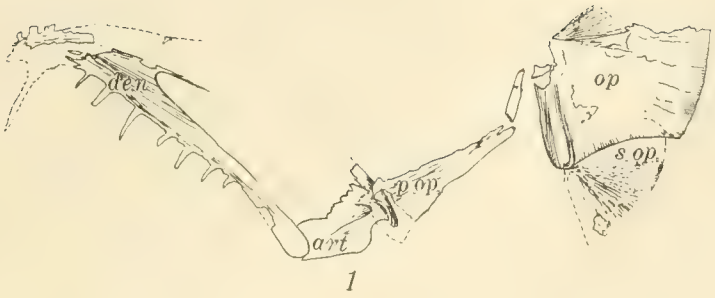
Enchodus amicrodus Stewart.

FIGURE 1.—*art*, articular; *den*, dentary; *op*, opercular; *sop*, subopercular; dentary and opercular are inverted. × ½. No. 51.

FIG. 2.—*den*, portion of dentary, inner view; *pt*, *pt'*, pterygoids; *qu*, *qu'*, quadrates; *pal*, *pal'*, palatines. × ½. No. 812.

FIG. 3.—Nearly complete skull: *art*, right articular; *art'*, posterior portion of left articular; *c-c*, contour of dorsal side of the skull, taken along the line, *s-s*; *c*, diagram of occipital condyle, the lower third being the basioccipital, and the upper two-thirds the exoccipitals; *ch*, ceratohyal; *ch'*, portion of the ceratohyal attached to the epihyal, which lies under the metapterygoid and is indicated by broken lines; *den*, right dentary; *den'*, impression of the left dentary in the chalk, showing that the symphysial portion was tuberculated externally; *eth*, ethmoid: *exo*, exoccipital; *fr*, frontal; *ms*, mesopterygoid; *mt*, metapterygoid; *op*, opercular; *pal*, *pal'*, palatines; *pop*, *pop'*, preoperculars; *pt*, *pt'*, pterygoids; *p*, parietal; *qu*, quadrate; *s*, suspensory pharyngeal; *sp*, sphenotic; *vt*, vertebra. × ½. No. 379.

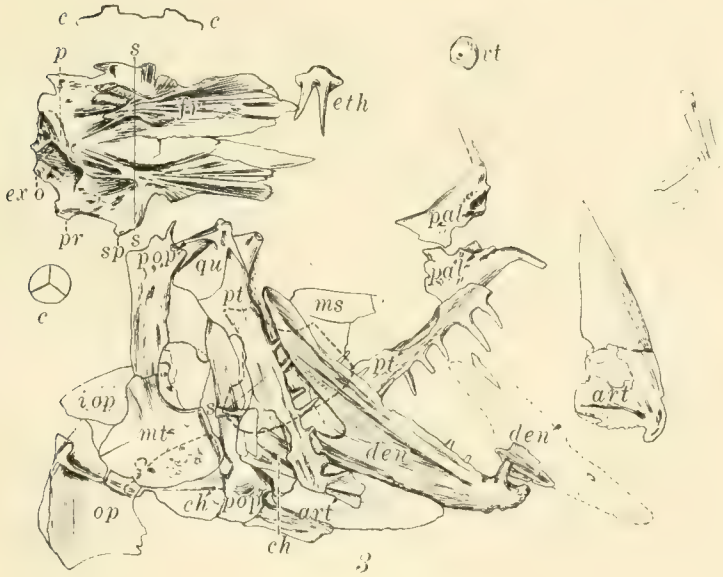
PLATE X.


den
op
s.op.
p.op.
art

1

pt
qu
pt
pal
qu
pal
den

2


c
c
p
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pt
ms
pal
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c
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mt
den
den
ch
pop
art
op
ch

3

PLATE XI.

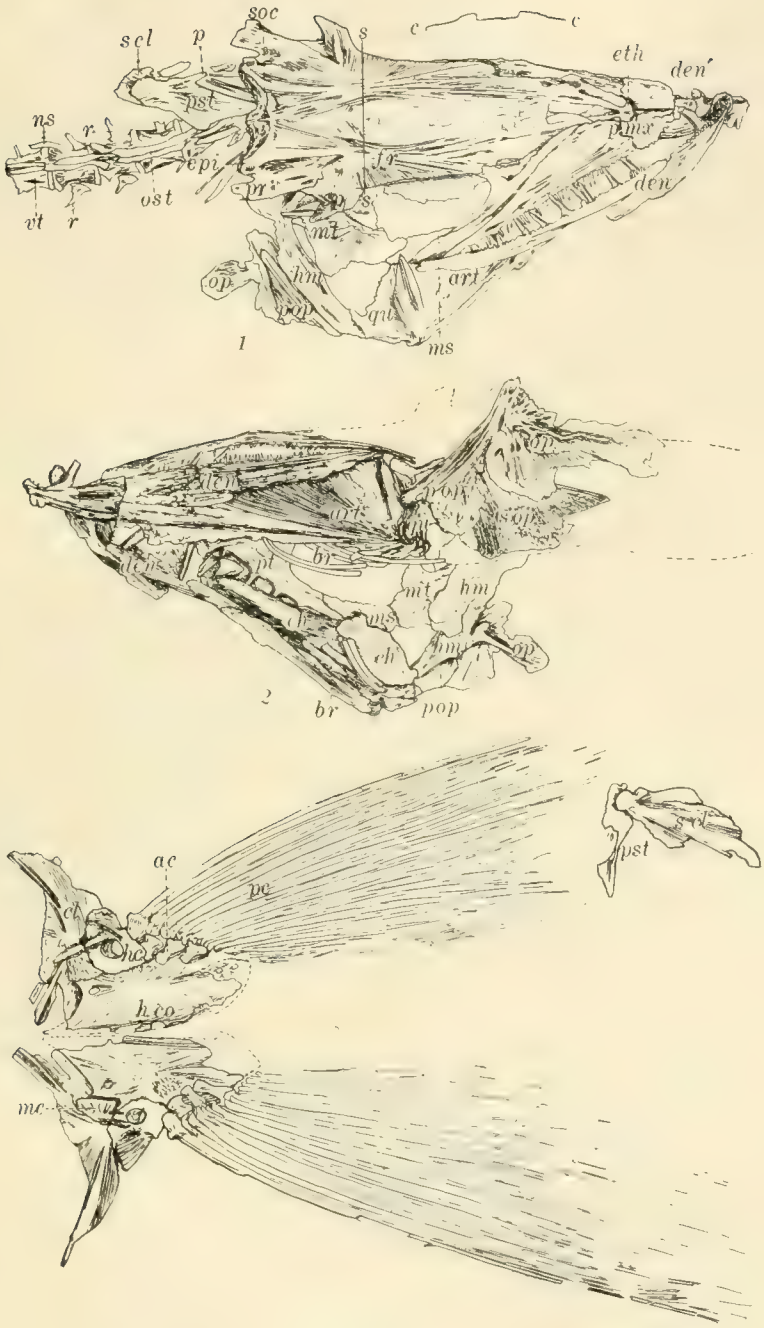
Enchodus dolichus Cope.

FIG. 1.—Complete skull, dorsal view: *art*, right articular; *c-c*, contour of dorsal side of skull along the line *s-s*; *den*, right dentary; *den'*, left dentary; *eth*, ethmoid; *fr*, right frontal; *hm*, hyomandibular; *ms*, mesopterygoid; *mt*, metapterygoid; *ns*, neural spine; *epi*, epiotic; *op*, opercular; *ost*, probably ossified tendons; *pmx*, premaxillary; *pop*, preopercular; *pr*, pterotic; *pst*, post-temporal; *qu*, right quadrate; *r*, proximal end of rib; *scl*, fragment of supraclavicle; *soc*, supraoccipital; *vt*, vertebra. × ½. No. 814.

FIG. 2.—Same as figure 1, ventral view: *art*, left articular; *br*, branchiostegal rays; *ch*, ceratohyal; *eh*, epihyal; *op*, left opercular; *pop*, left preopercular; *pt*, pterygoid; *qu*, left quadrate; *sop*, subopercular. Remainder of the lettering is the same as figure 1. × ½. No. 814.

FIG. 3.—Pair of pectoral fins and girdle apparatus belonging to the skull, figures 1 and 2: *ac*, actinosts; *cl*, clavicle; *hco*, hypocoracoid; *hc*, hypercoracoid; *mc*, mesocoracoid; *pc*, pectoral rays; *pst*, post-temporal; *scl*, supraclavicle. × ½. No. 814.

PLATE XI.


soc
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s
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pmx
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epi
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art
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1
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pop
2
ac
pc
ch
hco
mc
pst

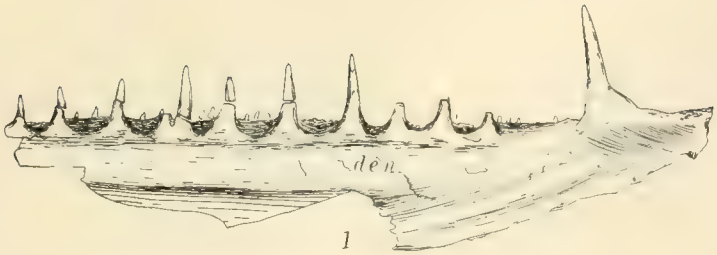
PLATE XII.

Enchodus dolichus Cope.

FIGURE 1.—Left dentary. × ½ (app). No. 820.

FIG. 2.—Part of skull and girdle bones, dissociated: *art*, articular; *art'*, cephalic end of articular; *br*, branchiostegals; *cl*, clavicle; *den*, left dentary; *den'*, right dentary; *hm*, hyomandibular; *iop*, interopercular; *mt*, probably the metapterygoid; *op*, left opercular; *op'*, right opercular; *pal*, right palatine; *pal'*, left palatine; *pop*, preopercular; *pt*, two pterygoid teeth; *sop*, subopercular. × ½ (ca). No. 174.

PLATE XII.



1

2

PLATE XIII.

Enchodus dolichus Cope.

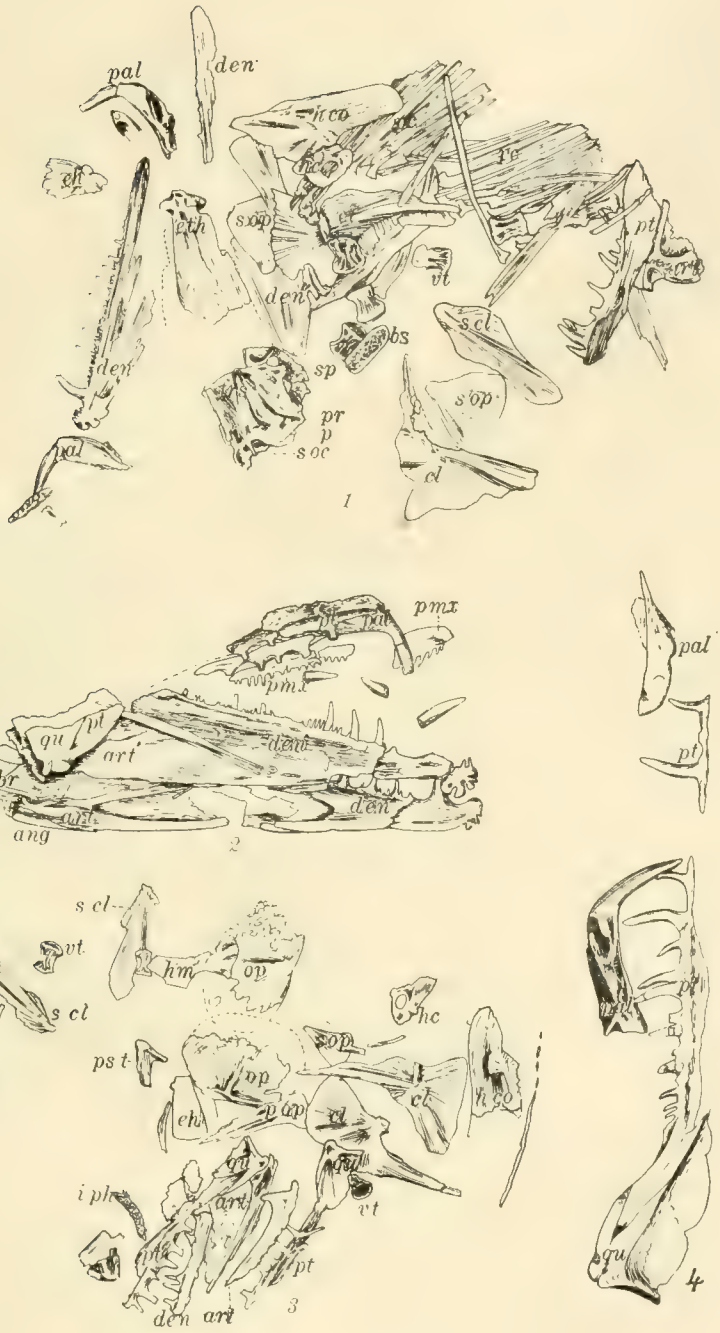
FIGURE 1.—Dissociated skull: *art*, articular; *bs*, probably a basibranchial; *cl*, clavicle; *den*, left dentary; *den'*, portion of right dentary; *den''*, portion of right dentary; *eh*, epihyal; *eth*, ethmoid; *hc*, hypercoracoid; *hco*, hypocoracoid; *pal*, left palatine; *pal'*, right palatine; *pc*, pectoral rays; *pr*, proötic; *p*, parietal; *pt*, pterygoid; *scl*, supraclavicle; *soc*, supraoccipital; *sop*, subopercular; *sp*, sphenotic. × ½. No. 808.

FIG. 2.—Bones of mandible: *art*, left articular; *art'*, right articular; *ang*, angular; *pal*, left palatine; *pt*, pterygoid; *qu*, quadrate. × ½. No. 816.

FIG. 3.—Dissociated skull: *hm*, hyomandibular; *iph*, probably an inferior pharyngeal; *op*, opercular; *pop*, preopercular. The remainder of the lettering is the same as in figure 1. × ½. No. 819.

FIG. 4.—Pair of palatines, a quadrate and pterygoid. The right palatine has its fang missing. × ½. No. 810.

PLATE XIII.


pal
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ect
ch
sop
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pmx
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scl
vt
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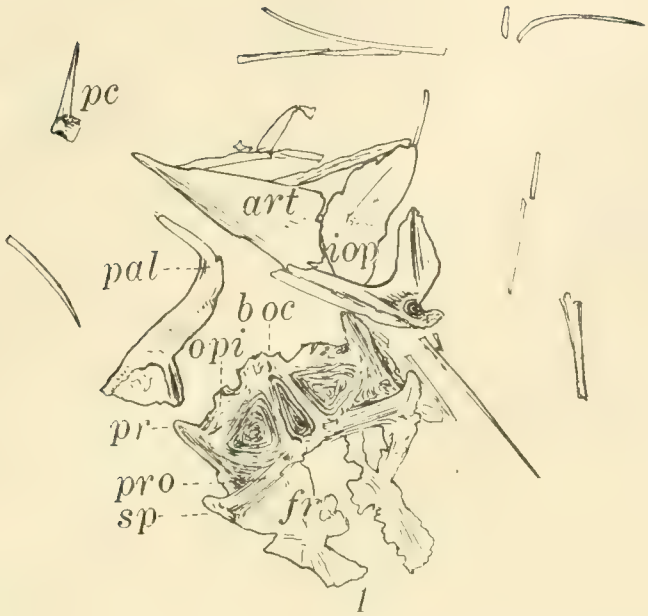
PLATE XIV.

Enchodus dolichus Cope.

FIGURE 1.—Posterior part of skull, ventral view: *art*, articular; *boc*, basioccipital; *fr*, frontal; *iop*, probably an interopercular; *opi*, opisthotic; *pal*, left palatine; *pc*, proximal end of a pectoral ray; *pr*, pterotic; *pro*, proötic; *sp*, sphenotic. × ½. No. 831.

FIGURE 2.—Mandibular elements: *art*, left articular; *art'*, right articular; *den*, right dentary; *den'*, left dentary; *hco*, hypocoracoid; *pal*, pair of palatines with fangs broken off; *pmx*, premaxillary; *pt'*, right pterygoid; *pt*, left pterygoid. × ½. No. 821.

PLATE XIV.


pc
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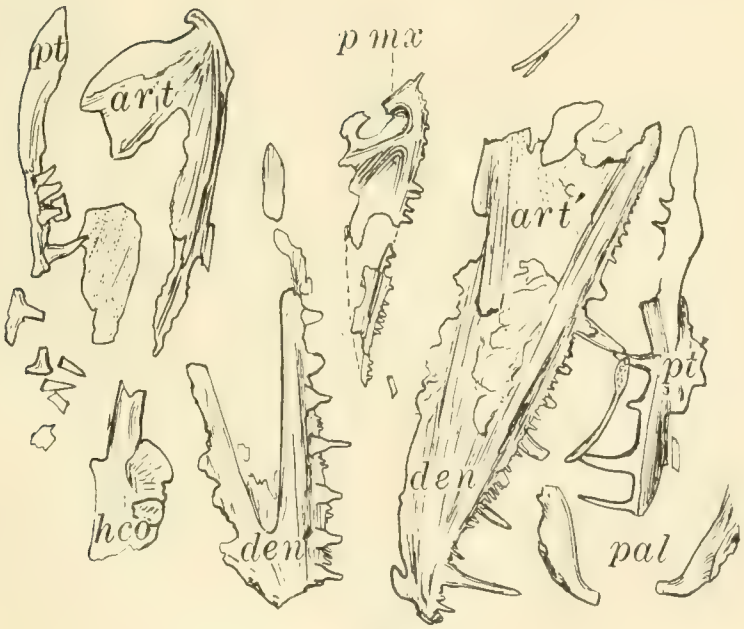
iop
pal
boc
opi
pr
pro
sp
fr

1


pt
art
p mx
art'
pt
den
hco
den
pal

2

PLATE XV.

Enchodus Shumardi Leidy.

FIGURE 1.—Nearly complete skull: *art*, articular; *br*, branchiostegals; *cl*, clavicle; *ch*, ceratohyal; *pop*, preopercular; *den*, dentary; *eth*, ethmoid; *fr*, frontal; *hm*, hyomandibular; *ms*, mesopterygoid; *mt*, metapterygoid; *op*, opercular; *pa*, parasphenoid; *pal*, palatine; *pmx*, premaxillary; *pt*, pterygoid; *qu*, quadrate; *r*, ribs; *sop*, subopercular. × ⅔. No. 829.

FIG. 2.—Nearly complete preopercular, *pop*, subopercular, *sop;* and fragment of opercular, *op*. × ⅔. No. 823.

FIG. 3.—Complete dentary. × ⅔. No. 824.

FIG. 4.—Fragmentary dentary. Type of *Enchodus parvus* Stewart. × ⅔. No. 321.

FIG. 5.—Fragmentary dentary. × ⅔. No. 825.

FIG. 6.—Fragmentary dentary. × ⅔. No. 826.

FIG. 7.—Fragmentary dentary. × ⅔. No. 323.

FIG. 8.—Palatine. × ¼. No. 838.

FIG. 9.—A palatine fang. × ⅔. No. 828.

FIG. 11.—*den*, right dentary; *den'*, left dentary; *pal*, palatine; *pmx*, premaxillary; *pop*, preopercular; *pt*, *pt'*, pterygoids; *qu*, quadrate; *sop*, subopercular. × ⅔. No. 827.

FIG. 13.—Impressions in chalk of a pair of palatines. × ⅔. No. 377.

Enchodus dolichus Cope.

FIGURE 10.—Mandible: *den*, dentary; *art*, articular. × ⅔. No. 156.

FIG. 12.—*den*, dentary; *pal*, right palatine; *pmx*, premaxillary. The left palatine lies on the upper edge of the premaxillary. × ⅔. No. 832.

Enchodus petrosus Cope.

FIGURE 14.—A palatine with the fang, showing the base of the discarded fang not yet absorbed. × ½. No. 807.

FIG. 15.—Ventral view of a palatine, showing the openings of the old blood cavities, on the crescentic scars of discarded fangs, and at the middle of the base of the last fang, which has been broken off. × ½. No. 824.

PLATE XV.

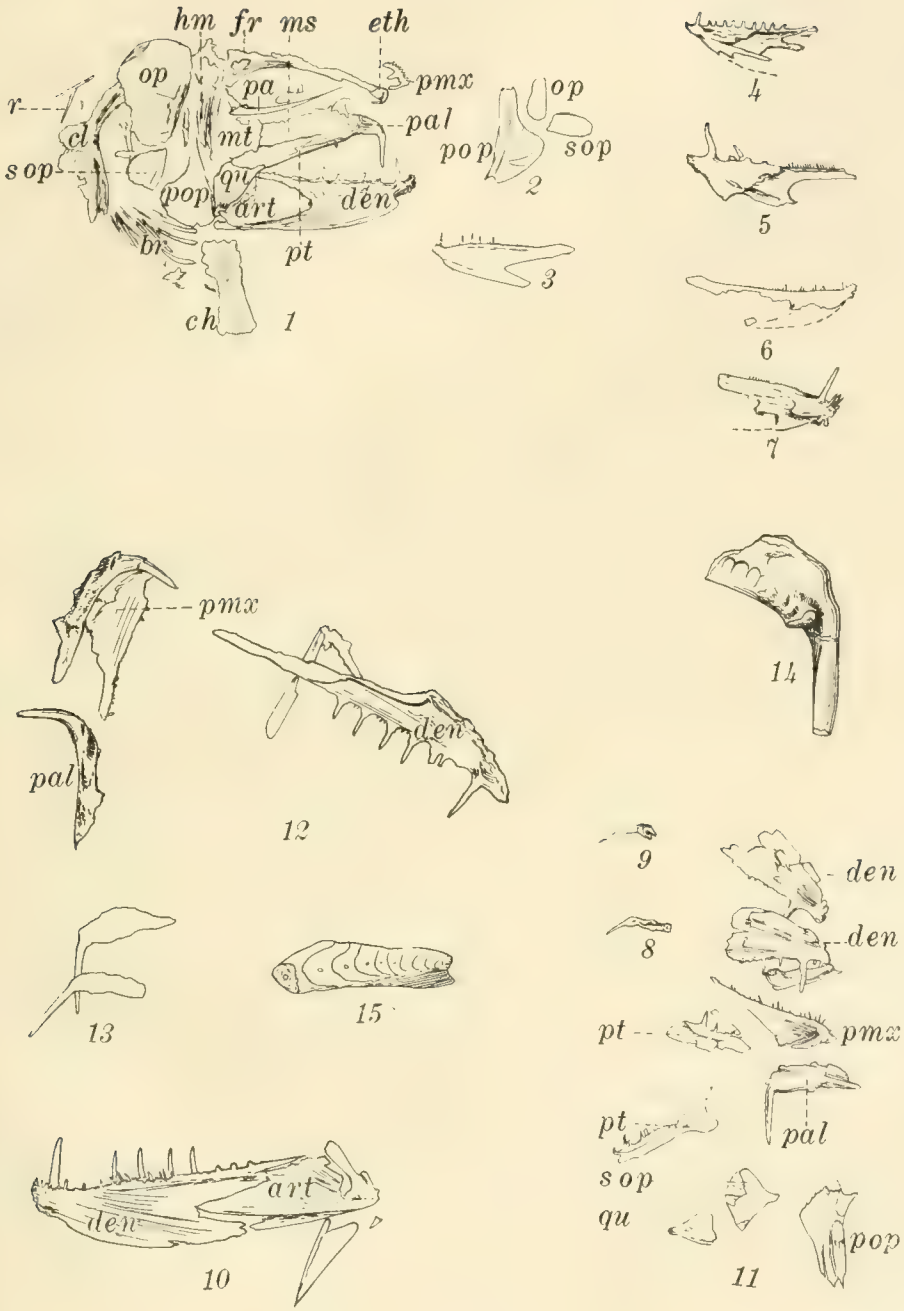

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PLATE XVI.

FIGURE 1.—Cross section of a palatine, showing the microscopic structure. The broken line *m-n* divides the true bone of the upper two-thirds from the osteodentine of the lower third; *a* and *b*, two large blood vessels of the osteodentine; *c*, a large blood vessel of the true bone. The osteodentine differs from the true bone above in being much more compact, more uniform in density, and in having fewer blood vessels. The mesial side of the section is to the right in the figure. The drawing was made by means of a camera lucida.

FIG. 2.—Diagram of a palatine bone. The line *x-x* indicates where the section (fig. 1) was taken.

PLATE XVI.

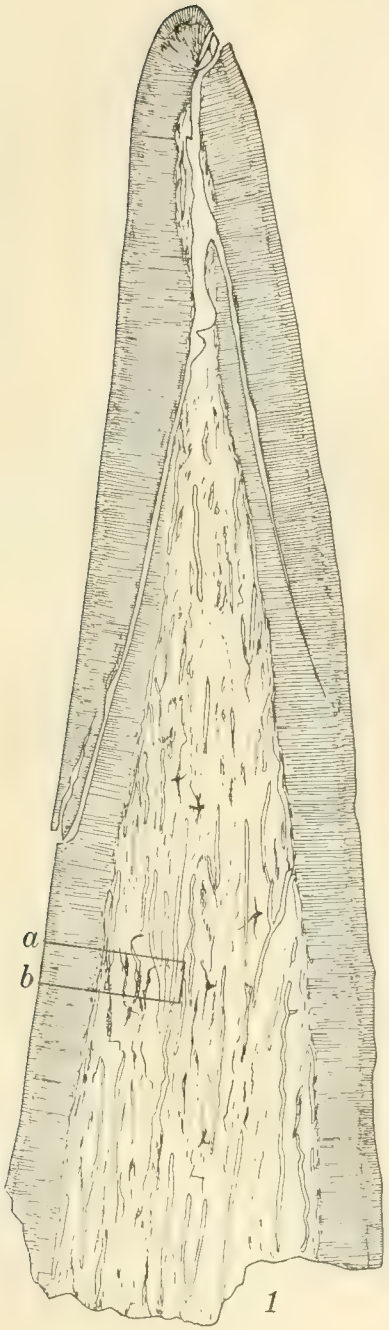
PLATE XVII.

FIGURE 1.—Longitudinal microscopic section of the distal end of the dental fang, from specimen No. 803. The small parallelogram, *a-b*, indicates where the enlarged section shown in figure 2 was taken. There is no pulp cavity, the interior of the tooth being composed of osteodentine very similar in structure to that of the palatine. An obscure line occurs near and parallel to the outer margin of the dentine. The fibers of the dentine are at right angles to the outer surface of the tooth.

FIG. 2.—A section of figure 1 more highly magnified: *e*, the enamel; *c*, the obscure dark line referred to above. In some places this line is hardly visible. Since there seems to be no difference in the structure of the dentine fibers at this point, it seems quite probable that it is caused by some peculiarity of the fossilization. *d*, dentine; *g*, patches of dentine in the osteodentine; *v*, blood vessel; *f*, bone cell cavity.

Both of these figures were drawn by means of a camera lucida.

PLATE XVII.



1

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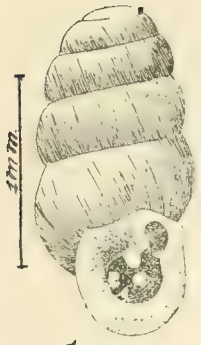
EXPLANATION OF PLATE XVIII.

1. *Bifidaria mcclungi.*
2. The same, showing the structure of the parietal tooth.
3. *Vertigo martini.*
4. *Vertigo gouldi.*
5. *Sphyradium hasta.*
6. *Sphyradium alticola.*

PLATE XVIII.



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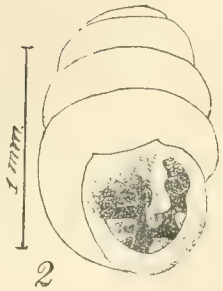

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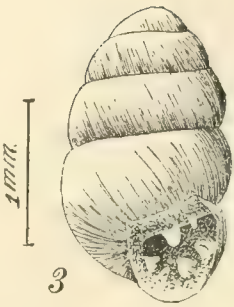

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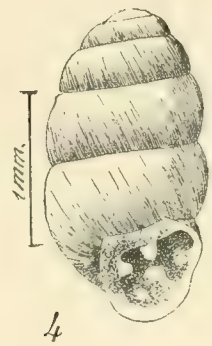

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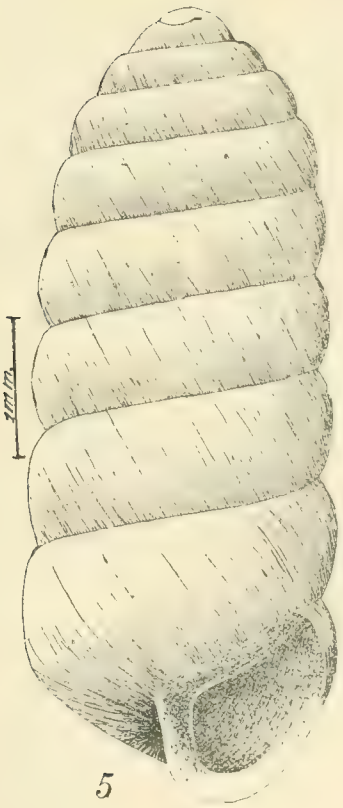

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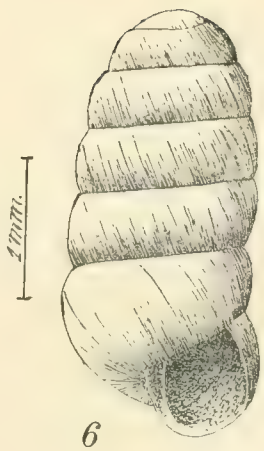

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EXPLANATION OF PLATE XIX.

act., acetabulum; *cir.*, cirrus; *cir. sac*, cirrus-sac; *ex. po.*, excretory pore; *ex. ves.*, excretory vesicle; *gen. po.*, genital pore; *in.*, intestinal cæcum; *La. can.*, Laurer's canal; *mo.*, mouth; *oe.*, œsophagus; *oa.*, ovary; *or.*, oral sucker; *ot.*, oötype; *ovi.*, oviduct; *p. prs.*, *pars prostatica; phx.*, pharynx; *phx. gl.*, pharyngeal gland; *sh. gl.*, shell gland; *tes.*, testis; *ves. sem.*, *vesicula seminalis; ut.*, uterus; *yo. gl.*, yolk gland; *yo. dt.*, yolk duct; *vag.*, metraterm.

All drawings were made with the aid of a camera lucida except figure 4.

FIG. 1.—*Renifer megasorchis;* ventral view. × 19.

FIG. 2.—*Renifer acetabularis;* ventral view. × 59.

FIG. 3.—*Renifer kansensis;* ventral view. × 20.

FIG. 4.—Diagram showing the female genital organs of R. *megasorchis.*

PLATE XIX.

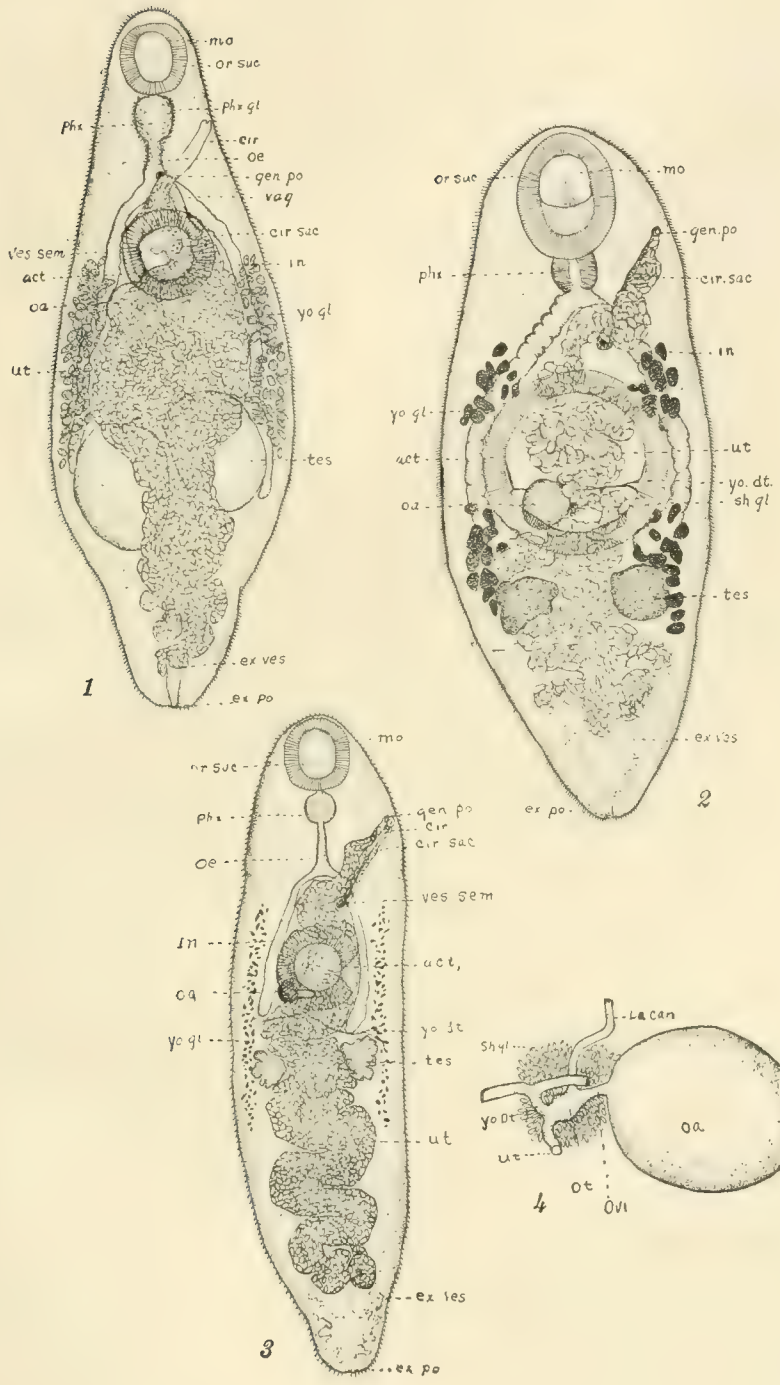
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PLATE XX.

Drawings made with a Zeichenprisma. Magnification · 2 16

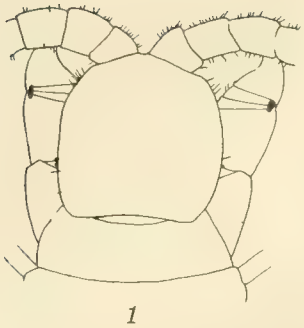
Arenophilus osborni, n. sp.

FIG. 1.—Dorsal view of anterior portion.
FIG. 2.—Ventral view of anterior portion.
FIG. 3.—Ventral view of posterior portion.

Geophilus dolichocephalus, n. sp.

FIG. 4.—Ventral view of posterior portion.
FIG. 5.—Ventral view of anterior portion.
FIG. 6.—Dorsal view of anterior portion.

PLATE XX.



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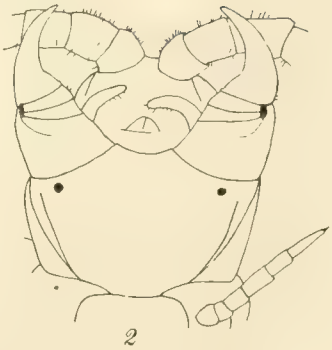

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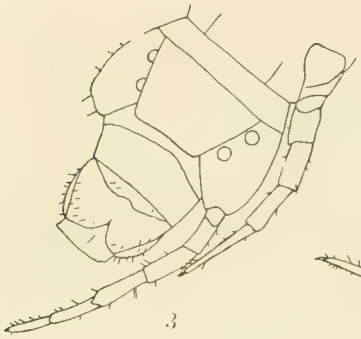

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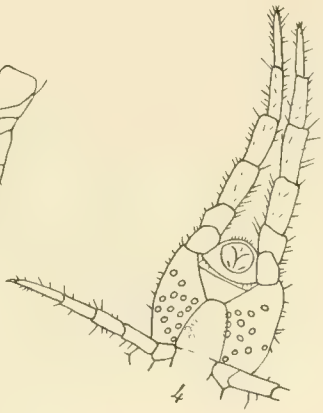

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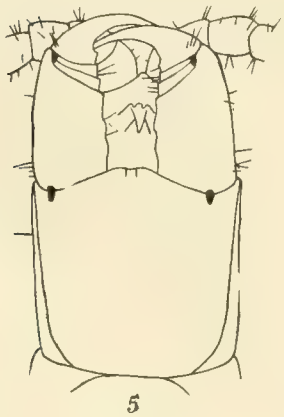

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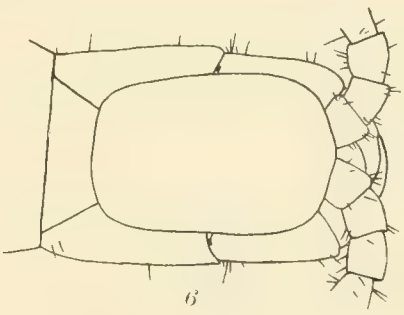

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PLATE XXI.

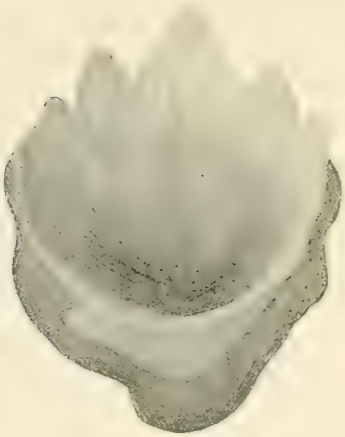


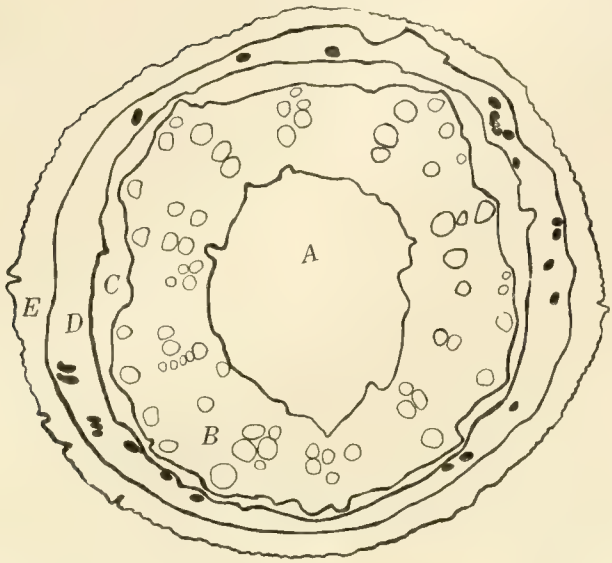
PLATE XXII.

Janusia gracilis.

FIG. 1. Cross section of young stem. × 42. *A*, pith; *B*, xylem; *C*. phloëm; *D*, pericycle; *E*, primary cortex.

FIG. 2. Cross section of old stem. × 40. *A*, pith; *B*, xylem; C. phloëm; *D*, parenchyma of the pericycle; *E*, the part of the cortex which contains the chlorophyll-bearing palisade cells, outside of which is the cork.

PLATE XXII.



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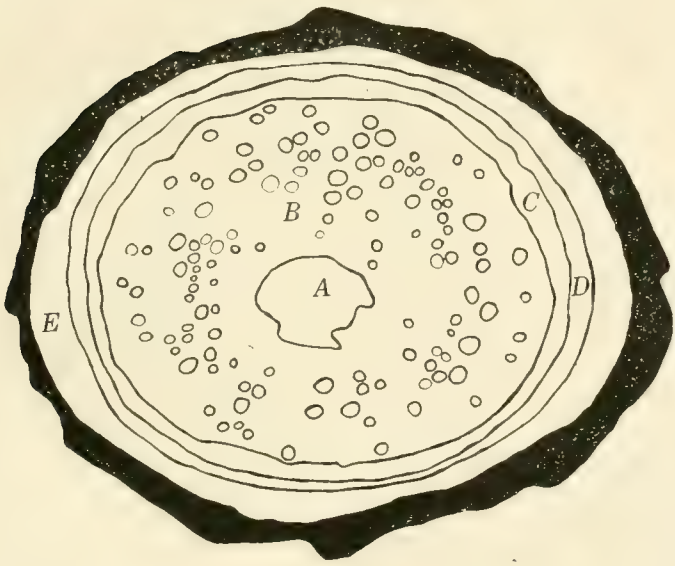

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PLATE XXIII.

FIG. 3. Wood fibers. × 417.

FIG. 4. Wood fibers. × 157.

FIG. 5. Wood fibers. × 157.

FIG. 6. Wood fiber. × 417.

FIG. 7. Wood fiber. × 335.

FIG. 8. Part of bast fiber. × 362.

FIG. 9. Bast fiber. × 5.

FIG. 10. Cross section of stem, showing scattered bast fibers. × 356.

FIG. 11. Part of cross section of stem. × 335.

FIG. 12. Epidermis, parenchyma, and bast fibers of young stem. × 335. *A*, pore in epidermis.

PLATE XXIII.

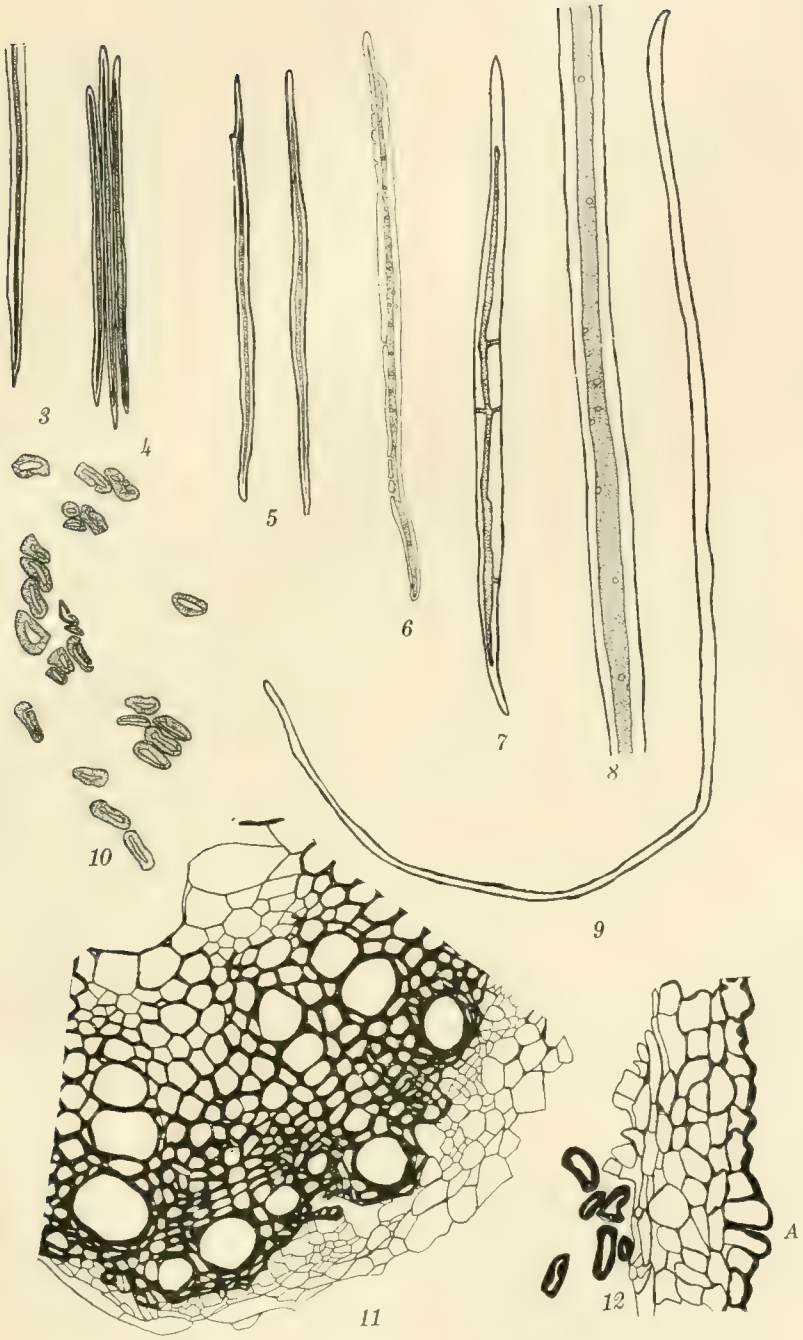


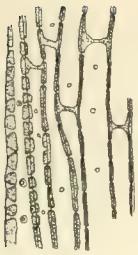
PLATE XXIV.

FIG. 13. Wood parenchyma cells and fiber tracheids. × 295.

FIG. 14. Longitudinal section of young stem, showing tracheal tubes and wood parenchyma. × 335.

FIG. 15. Longitudinal section of part of old stem. × 335. *A*, phloëm parenchyma cells; *B*, parenchyma cells of the pericycle; *C*, suberised cells.

PLATE XXIV.



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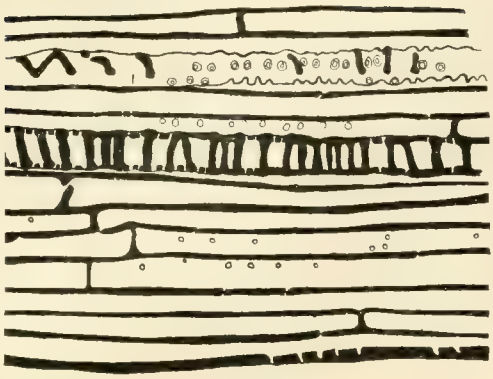

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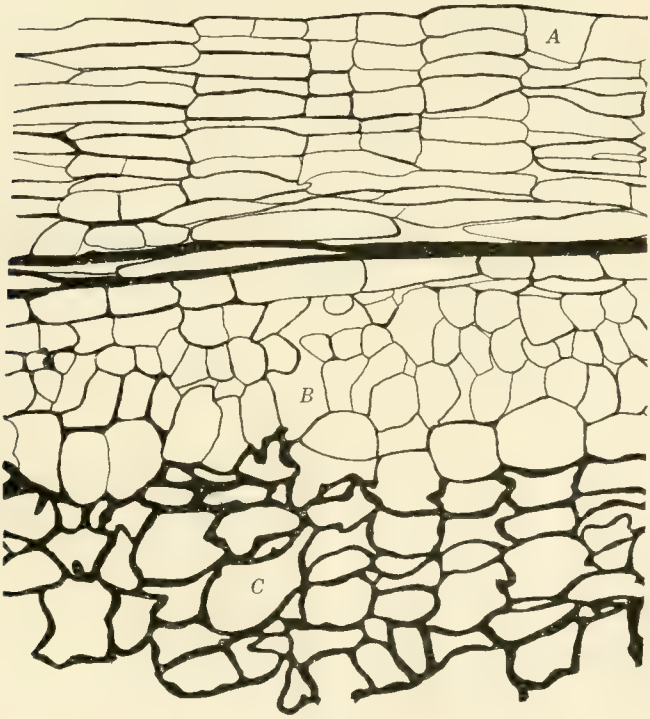

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PLATE XXV.

FIG. 16. Portion of xylem with tracheal tubes. × 295.

FIG. 17. Surface view of epidermis of stem. × 295.

FIG. 18. Pitter tracheal tubes. × 335.

FIG. 19. Medullary ray cells as seen in longitudinal section. × 630.

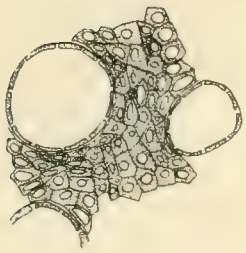
FIG. 20. Epidermis of stem. × 315.

FIG. 21. Pith cells. × 295.

FIG. 22. Palisade cells as found in the pericycle of old stems. × 280. *A*, cork; *B*, parenchyma.

FIG. 23. Stoma of stem. × 315.

PLATE XXV.



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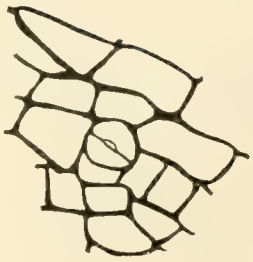

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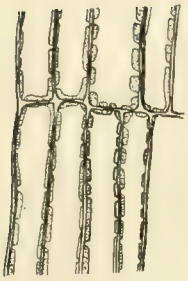

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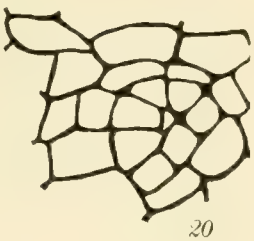

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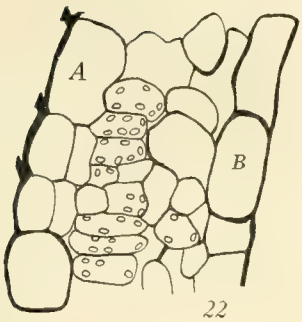

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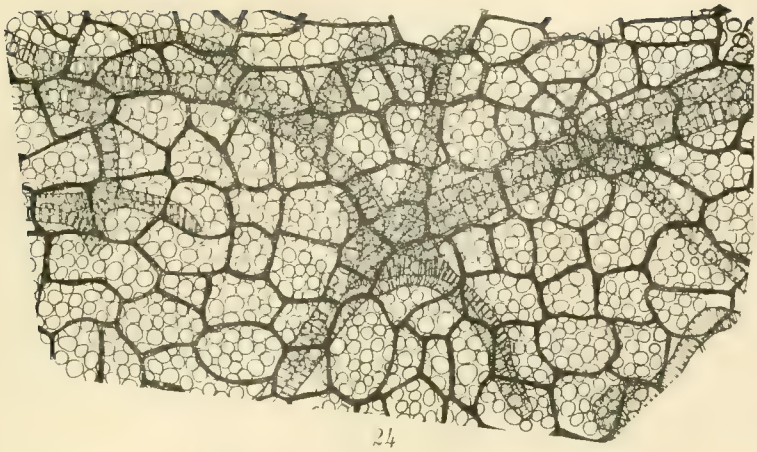
PLATE XXVI.

FIG. 24. Surface view of leaf, showing upper epidermis, palisade cells, and veins. × 335.

FIG. 25. Leaf with complete venation. × 10.

FIG. 26. Surface of part of a leaf, showing the hairs as they appear. × 40.

PLATE XXVI.



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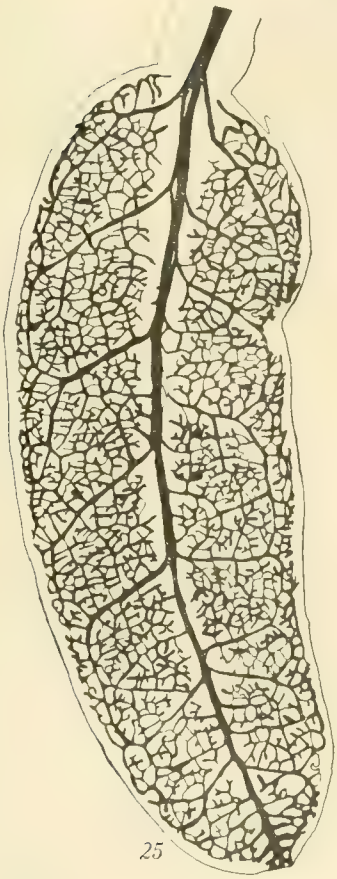

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PLATE XXVII.

FIG. 27. Cross section of leaf. × 270.

FIG. 28. Cross section of pore found in leaf. × 337.

FIG. 29. Longitudinal section of leaf, showing hair in place. × 350. *A*, cutinized part of stalk of trichome; *B*, continuation of outer wall of the epidermis; *C*, pore.

FIG. 30. Part of longitudinal section of leaf hair. × 295.

FIG. 31. Stem hairs. × 80.

PLATE XXVII.

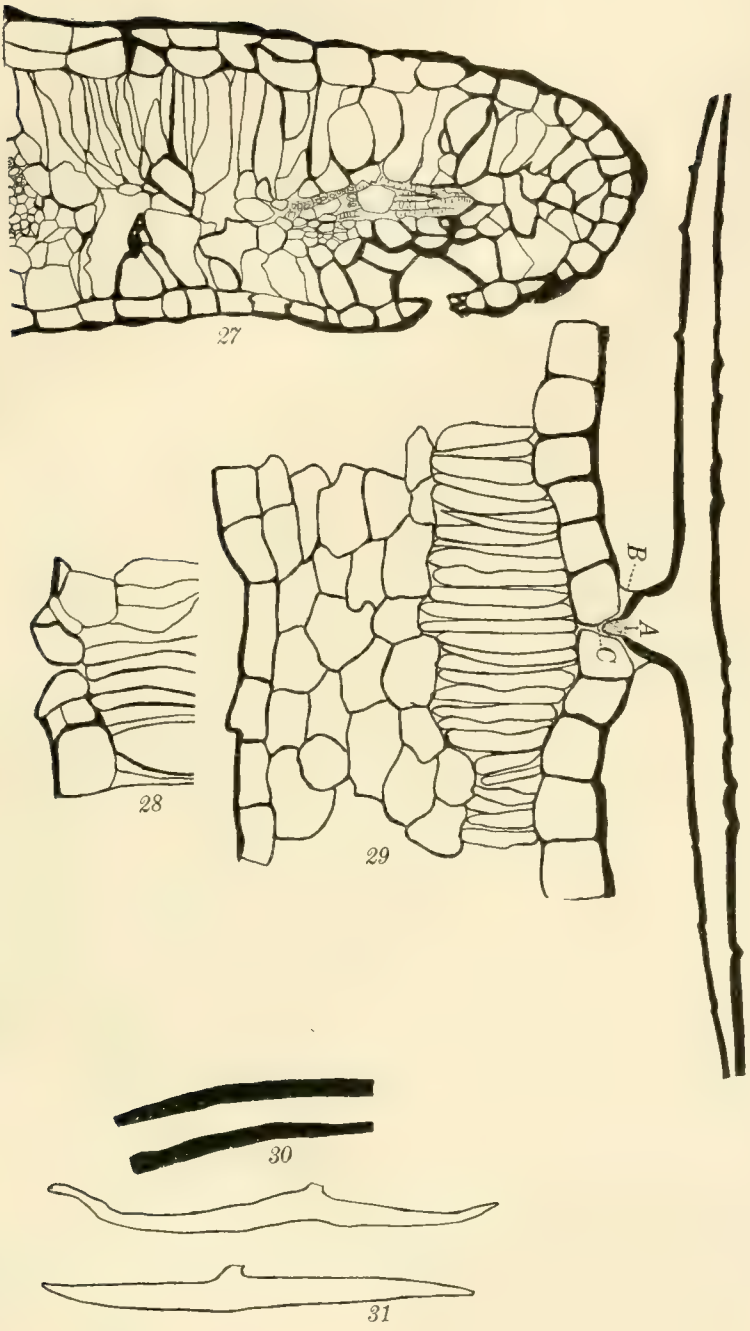
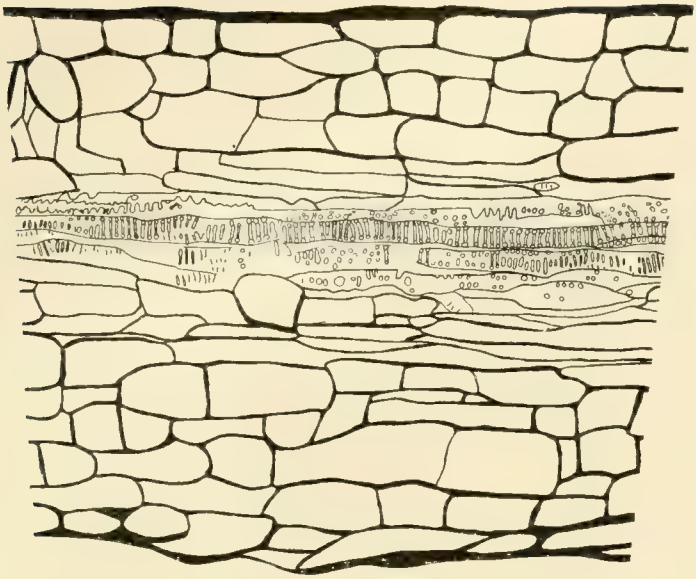


PLATE XXVIII.

FIG. 32. Longitudinal section of leaf through midrib. × 335.

FIG. 33. Cross section of leaf through midrib. × 335.

PLATE XXVIII.



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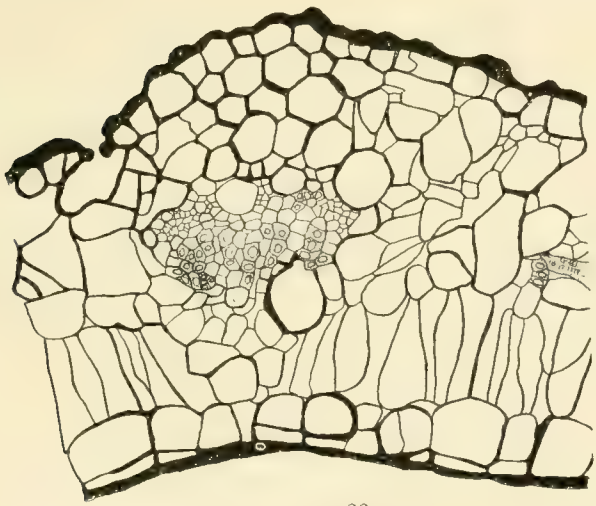

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PLATE XXIX.

page

FIG. 1. Difflugia pyriformis 202
FIG. 2. Difflugia lobostoma 202
FIG. 3. Difflugia pyriformis 202
FIG. 4. Difflugia lobostoma 202
FIG. 5. Difflugia pyriformis 202
FIG. 6. Difflugia pyriformis 202
FIG. 7. Difflugia globulosa 202
FIG. 8. Difflugia globulosa 202
FIG. 9. Difflugia pyriformis? (much like *D. globulosa*) 202
FIG. 10. Difflugia pyriformis 202
FIG. 11. Difflugia arcula 203

PLATE XXIX.

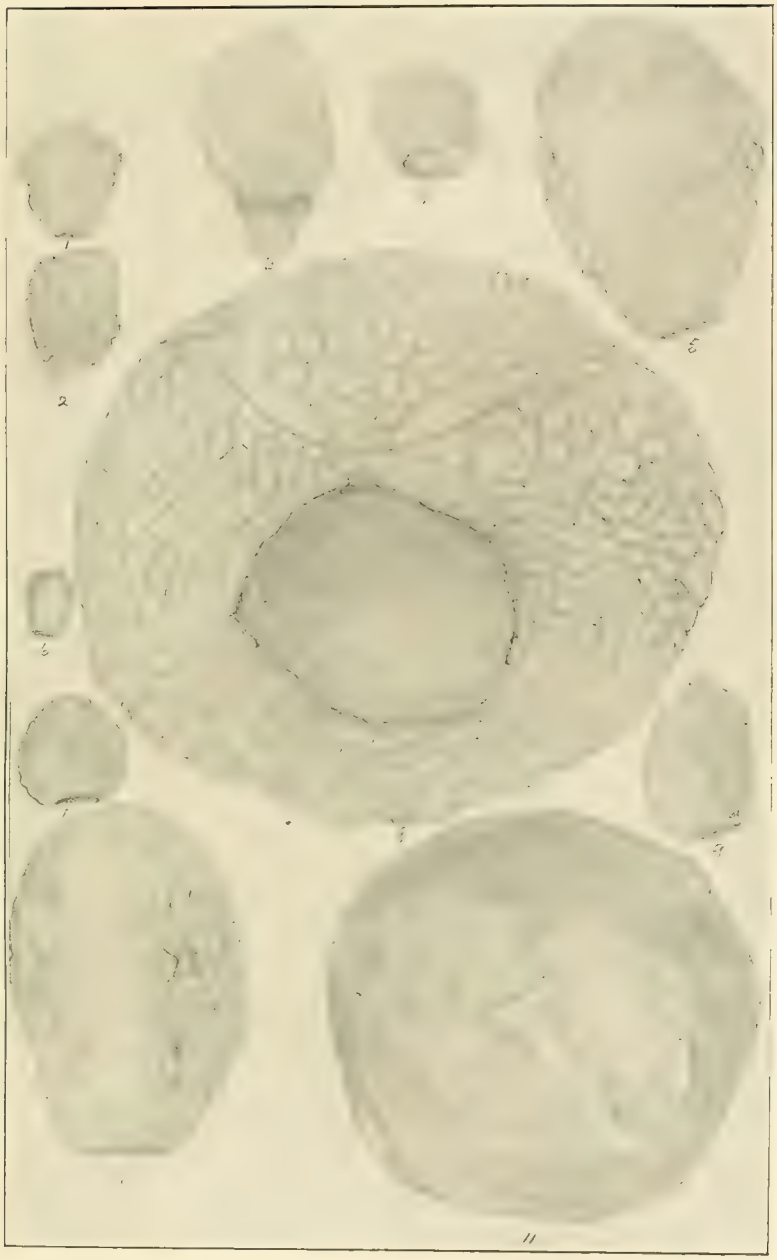

PLATE XXX.

page

FIG. 12. Difflugia corona 202
FIG. 13. Difflugia globulosa 202
FIG. 14. Difflugia globulosa (?) 202
FIG. 15. Difflugia constricta 203
FIG. 16. Difflugia globulosa 202
FIG. 17. Difflugia globulosa 202

PLATE XXX.

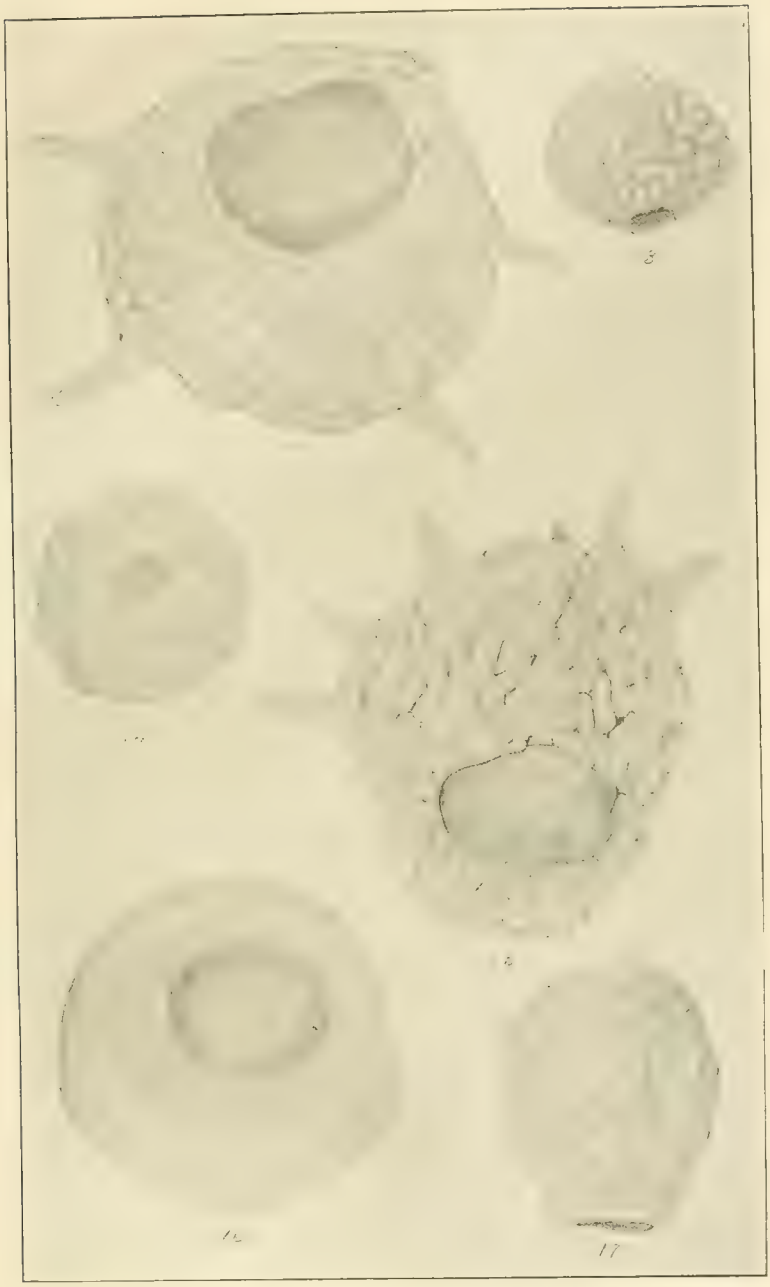

PLATE XXXI.

		page
Fig. 18.	Difflugia acuminata	203
Fig. 19.	Difflugia pyriformis	202
Fig. 20.	Difflugia globulosa	202
Fig. 21.	Difflugia globulosa	202
Fig. 22.	Difflugia pyriformis (conjugating)	202
Fig. 23.	Difflugia constricta—oral view	203
Fig. 24.	Difflugia constricta—lateral view of 23	203
Fig. 25.	Difflugia urceolata	203

PLATE XXXI.

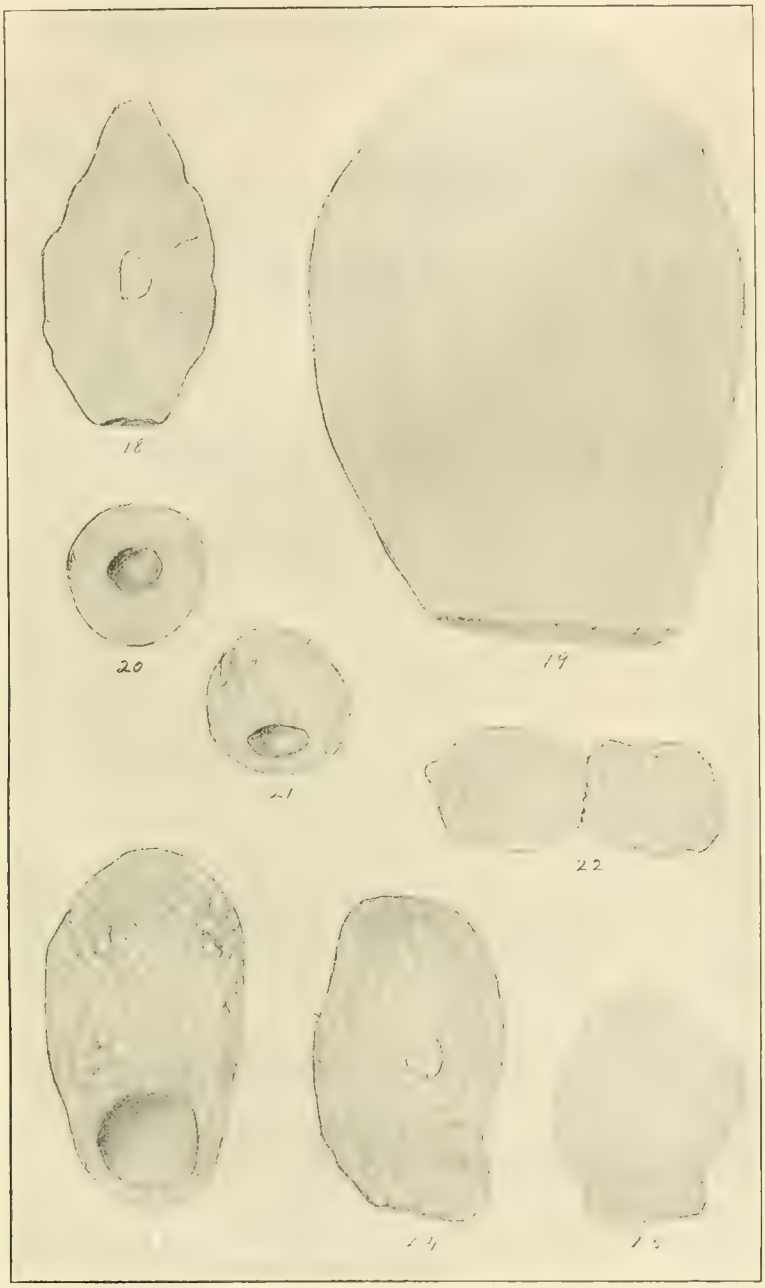

PLATE XXXII.

page

FIG. 26. Difflugia acuminata 203
FIG. 27. Difflugia urceolata 203
FIG. 28. Difflugia pyriformis 202
FIG. 29. Difflugia globulosa 202
FIG. 30. Difflugia globulosa 202
FIG. 31. Difflugia pyriformis 202
FIG. 32. Difflugia globulosa 202
FIG. 33. Difflugia lobostoma 202
FIG. 34. Pseudodifflugia gracilis 203
FIG. 35. Pseudodifflugia gracilis 203
FIG. 36. Pseudodifflugia gracilis 203

PLATE XXXII.

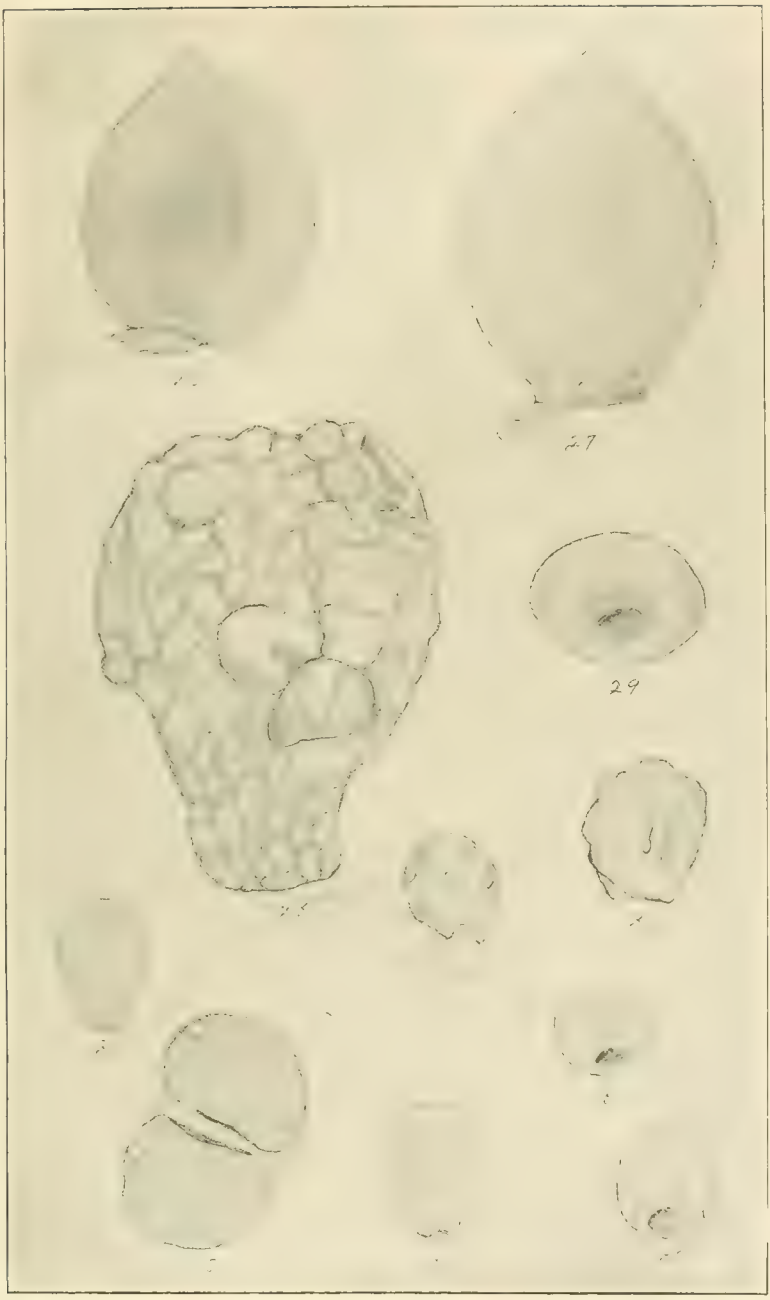

PLATE XXXIII.

page

FIG. 37. Arcella vulgaris, oral view 204
FIG. 38. Arcella vulgaris, side view 204
FIG. 39. Arcella vulgaris, dorsal view 204
FIG. 40. Arcella vulgaris, tipped 204
FIG. 41. Arcella vulgaris, oral view 204
FIG. 42. Arcella discoides, oral view 204
FIG. 43. Arcella discoides, side view 204
FIG. 44. Arcella vulgaris, side view 204
FIG. 45. Arcella vulgaris, oral view 204
FIG. 46. Arcella vulgaris, side view 204
FIG. 47. Arcella vulgaris, top view 204

PLATE XXXIII.

PLATE XXXIV.

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FIG. 48. Arcella vulgaris, oral view 204
FIG. 49. Arcella vulgaris, top view 204
FIG. 50. Arcella vulgaris, side view 204
FIG. 51. Trinema euchelys 205
FIG. 52. Trinema euchelys 205
FIG. 53. Trinema acinus 205
FIG. 54. Trinema euchelys, encysted 205
FIG. 55. Trinema euchelys 205
FIG. 56. Euglypha alveolata 204
FIG. 57. Euglypha ciliata 204
FIG. 58. Euglypha alveolata 204
FIG. 59. Assulina seminulum 204

PLATE XXXIV.

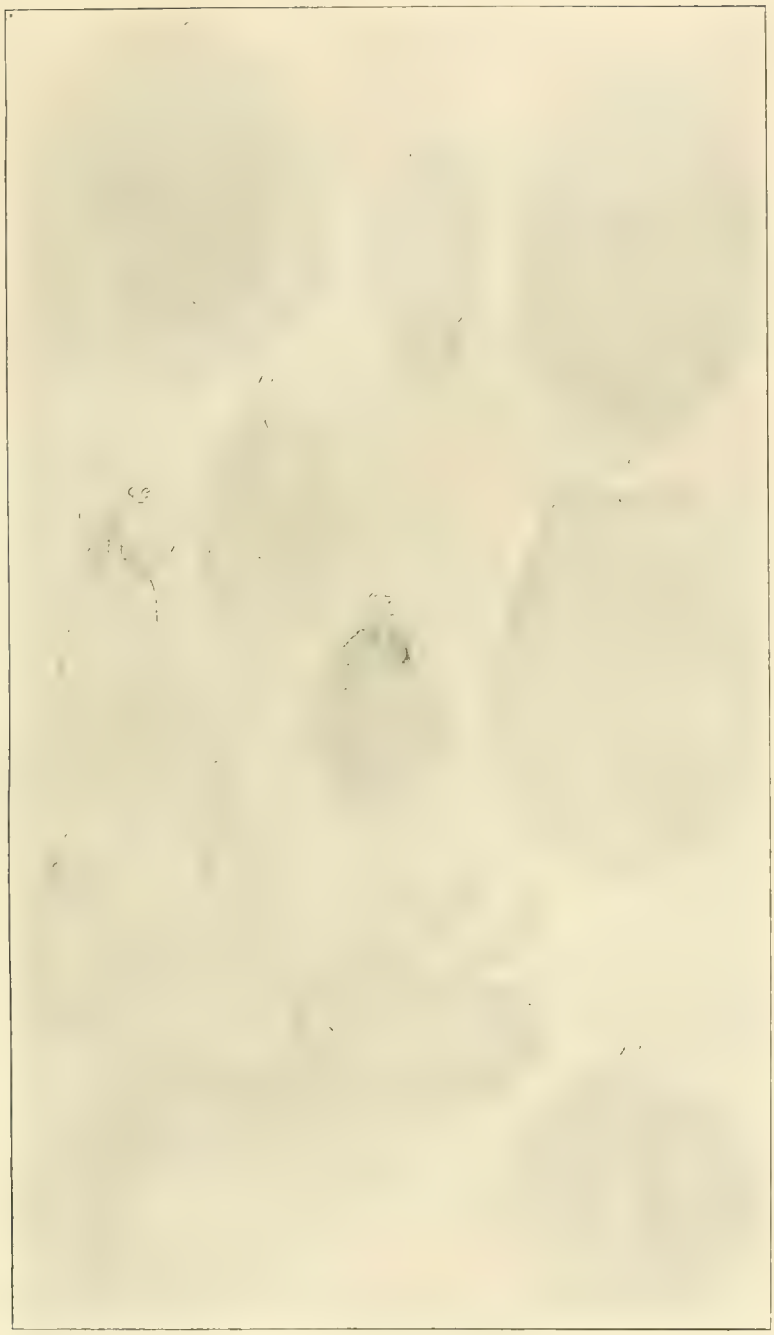

PLATE XXXV.

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Fig. 60.	Centropyxis aculeata	205
Fig. 61.	Centropyxis ecornis	205
Fig. 62.	Heleopera petricola	205
Fig. 63.	Hyalosphenia cuneata	205

PLATE XXXV.

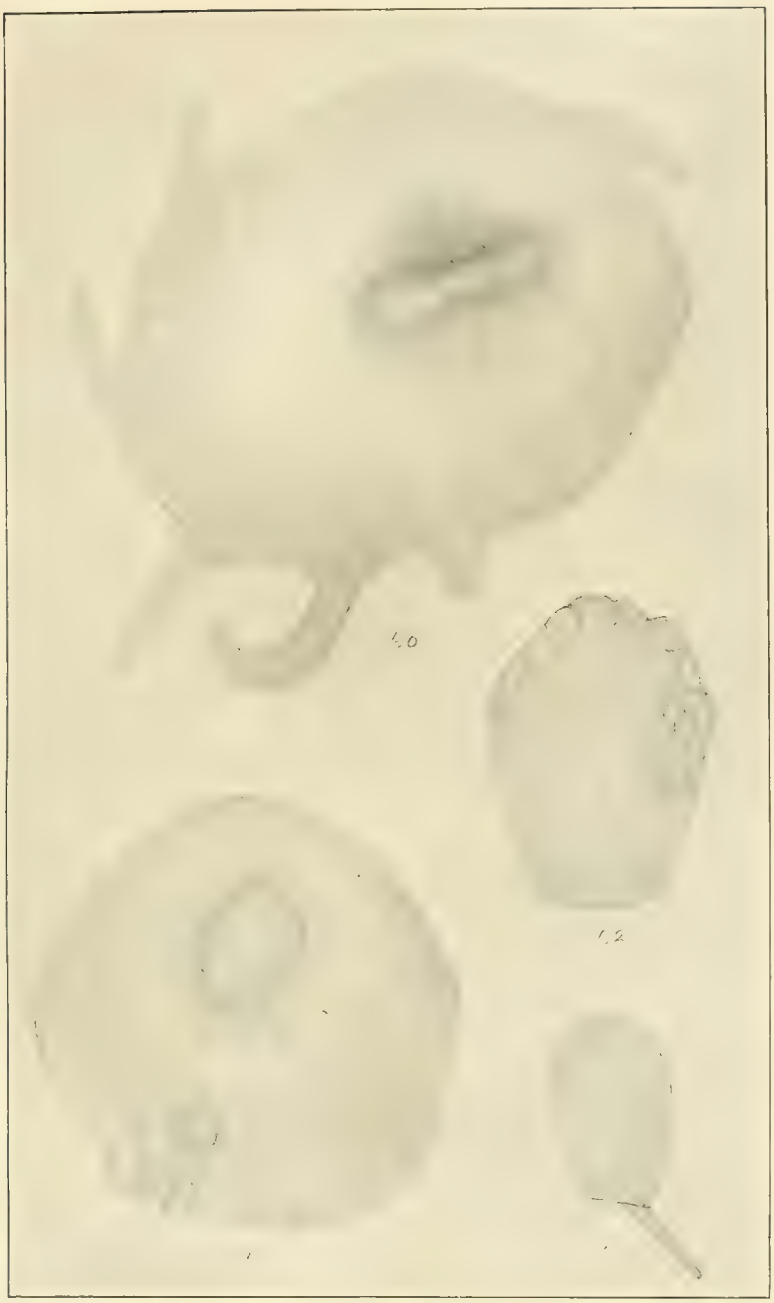

PLATE XXXVI.

(No. 9.)

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Fig. 64.	Hyalosphenia papillio	205
Fig. 65.	Pamphagus mutabilis	205
Fig. 66.	Pamphagus mutabilis	205
Fig. 67.	Pamphagus mutabilis	205
Fig. 68.	Pamphagus hyalinus	206
Fig. 69.	Nebela collaris	206

PLATE XXXVI.

(No. 9.)

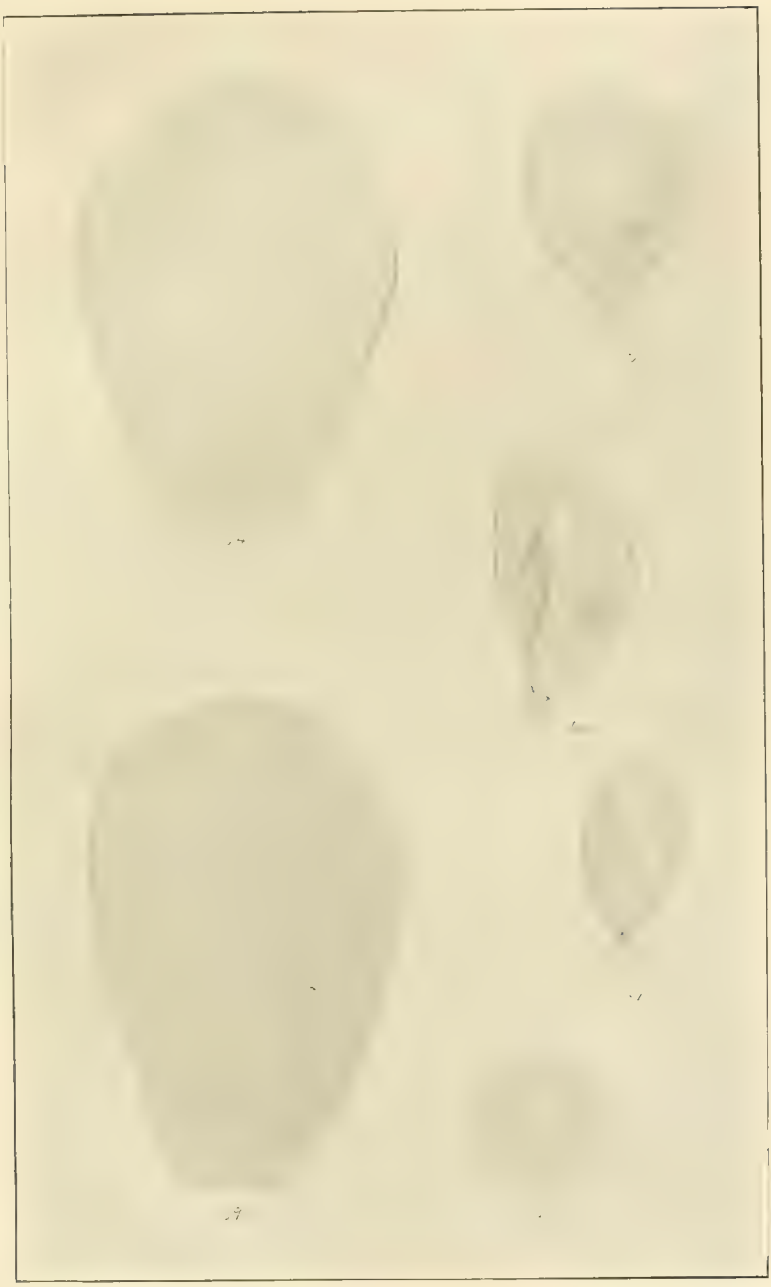

PLATE XXXVI.

(No. 15.)

FIG. 1. Cross section of old stem. × 17½. *a*, collenchyma rib; *b*, calcium oxalate crystals; *c*, palisade parenchyma cells; *d*, tracheal tubes; *e*, phloëm masses surrounded by xylem; *f*, calcium oxolate crystals.

FIG. 2. Cross section of young stem. × 34. *a*, short chloroplast-bearing cells; *b*, secondary xylem mass; *c*, palisade parenchyma.

FIG. 3. Cross section of root. × 78. *a*, large parenchyma cells, aqueous tissue; *b*, small parenchyma cell of the pericycle; *c*, masses of phloëm; *d*, tracheal tubes.

PLATE XXXVI.

(No. 15.)

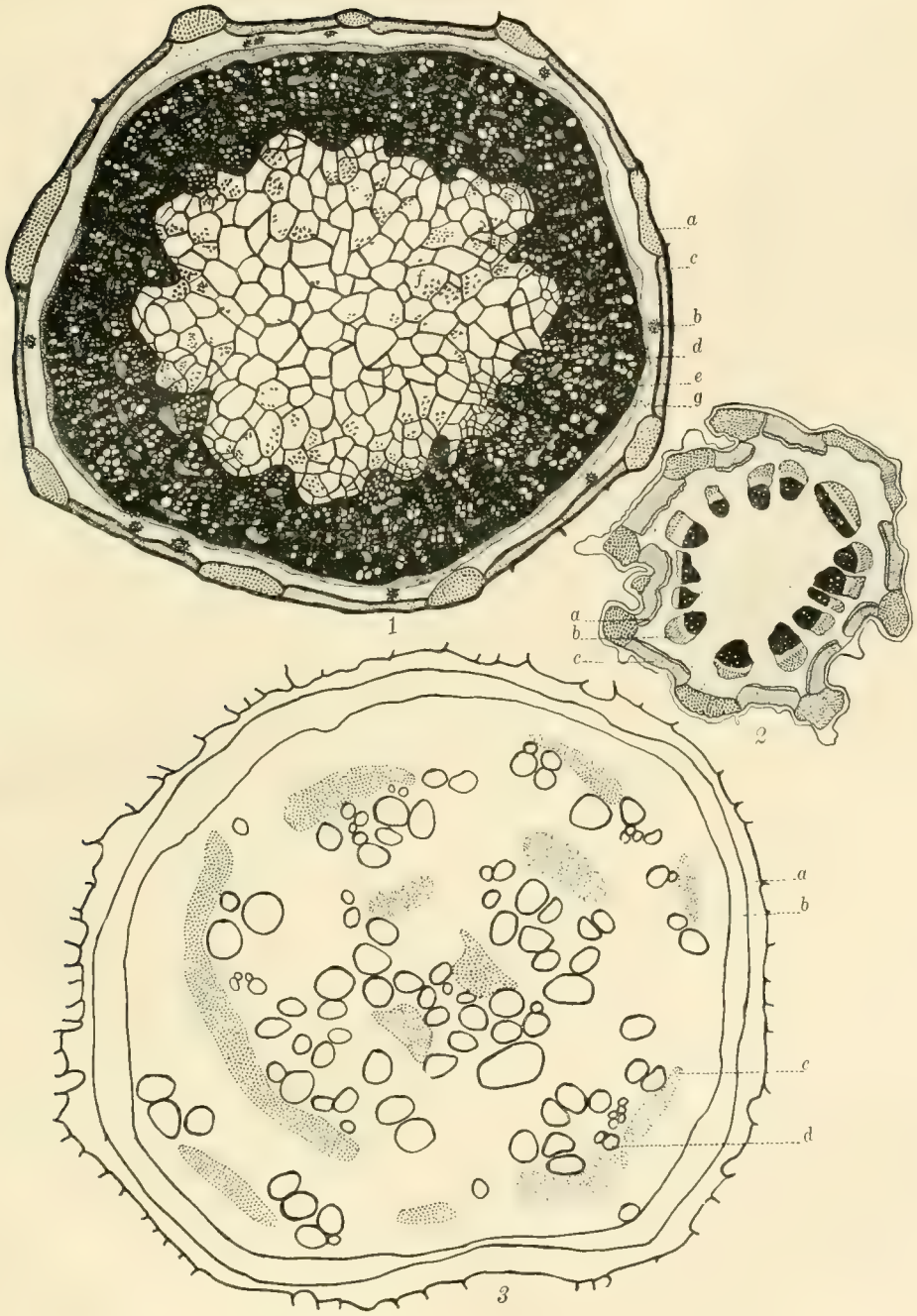

PLATE XXXVII.

FIG. 4. Primary vascular bundle. × 375.

FIG. 5. Section of cross section of old stem. × 100. *a*, epidermis; *b*, palisade parenchyma; *c*, short chloroplast-bearing cells; *d*, aqueous tissue; *e*, phloëm; *f*, collenchyma rib.

FIG. 6. Primary vascular bundle from young stem. × 995.

FIG. 7. Phloëm from primary vascular bundle of old stem. × 450. *a*, protein grains.

PLATE XXXVII.

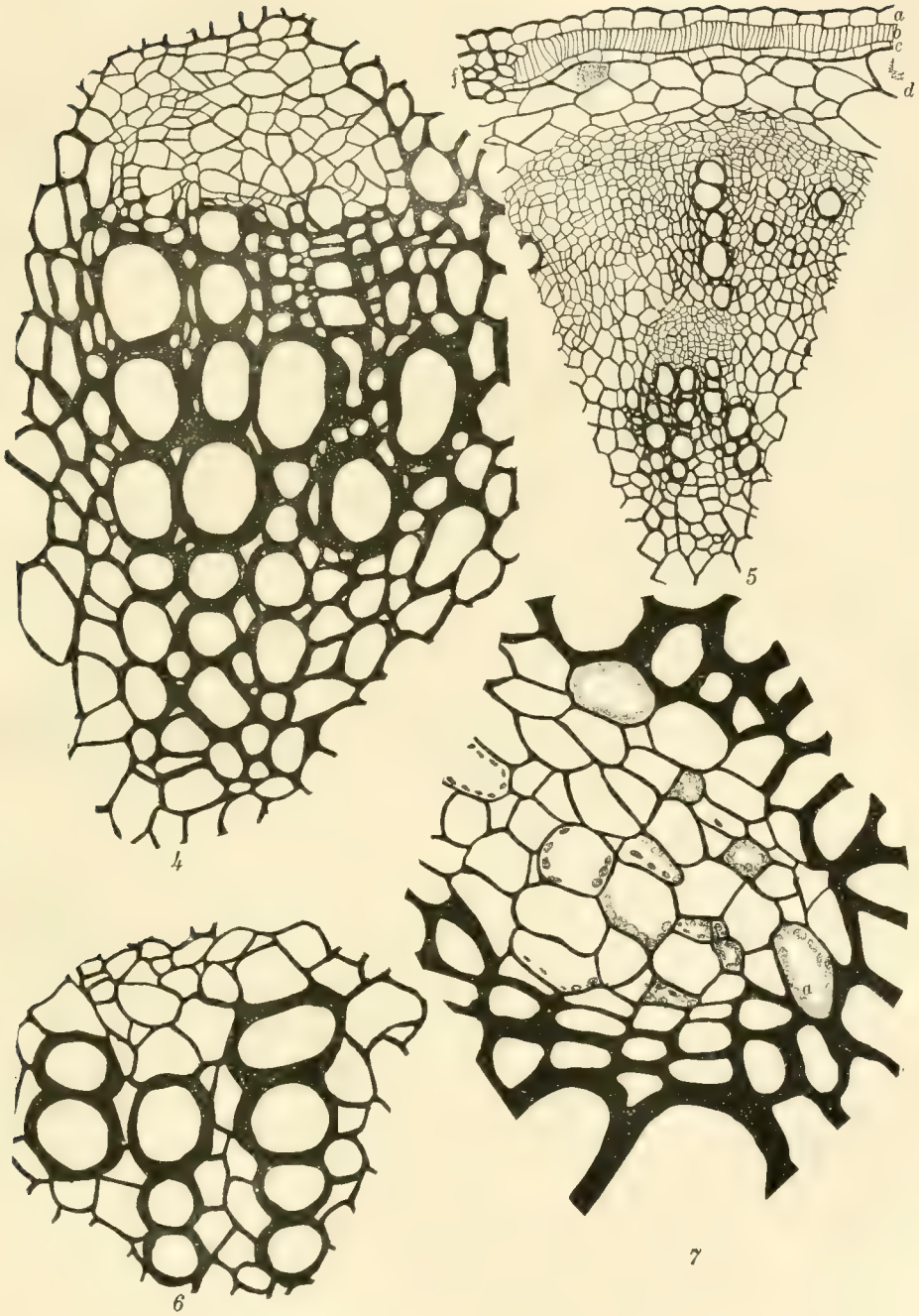

PLATE XXXVIII.

FIG. 8. Cross section of collenchyma rib. × 255.

FIG. 9. Longitudinal section of collenchyma rib. × 165. *a*, collenchyma cells; *b*, parenchyma cells.

FIG. 10. Cross section of stoma. × 285.

FIGS. 11 AND 12. Trichomes from stem. × 55. *a*, basil cell.

FIG. 13. Epidermis from stem. × 183. *a*, epidermal cells above collenchyma ridge; *b*, epidermal cells in channels.

FIG. 14. Longitudinal section of stem. × 165. *a*, guard cell of stoma; *b*, palisade parenchyma; *c*, short chloroplast-bearing cells; *d*, parenchyma cell of an aqueous tissue.

PLATE XXXVIII.

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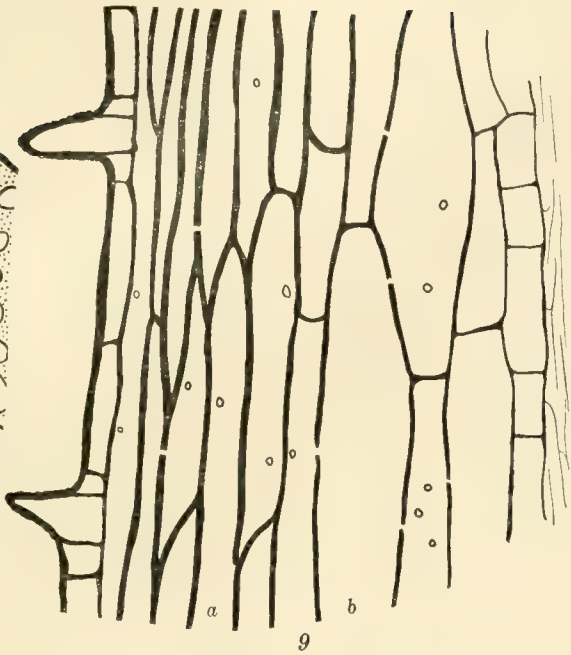

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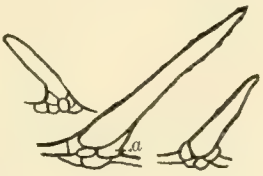

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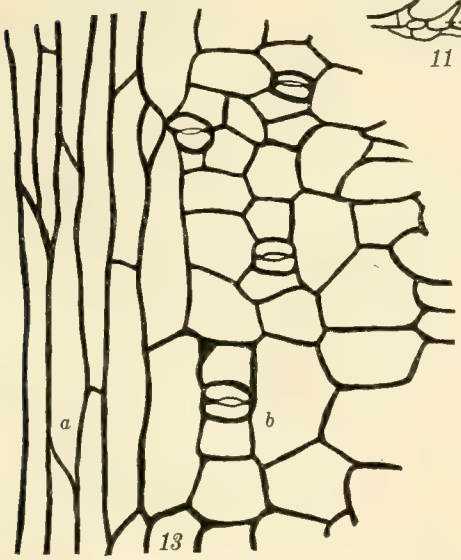

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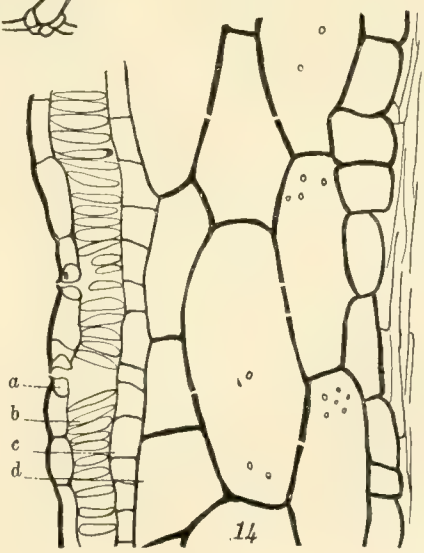


14

PLATE XXXIX.

FIG. 15. Wood fibers. × 230.

FIG. 16. Cross section in stem. × 390. *a*, phloëm; *b*, parenchyma of pericycle; *c*, starch sheath; *d*, starch; ➔, center.

FIG. 17. Compound grains of starch. × 995.

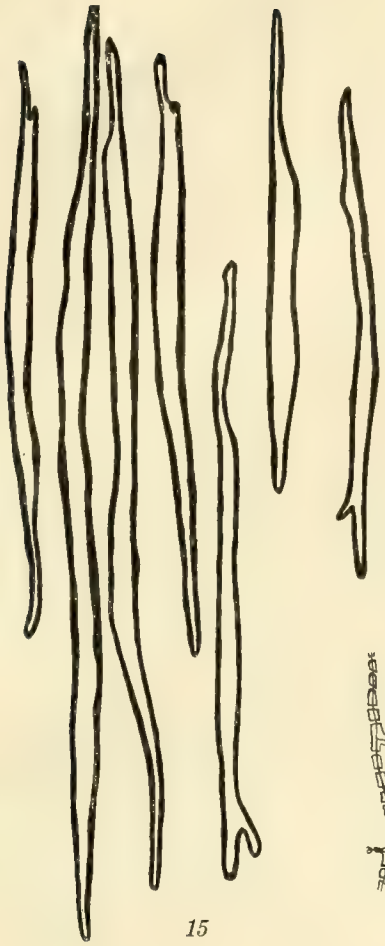
FIG. 18. Xylem parenchyma. × 255.

FIG. 19. Longitudinal section of secondary phloëm × 255. *a*, calcium oxalate crystals.

FIG. 20. Longitudinal section of stem; primary bundles. × 390. *a*, xylem parenchyma; *b*, pith.

FIG. 21. Tracheids and tracheæ; mascerated sections. × 390.

PLATE XXXIX.



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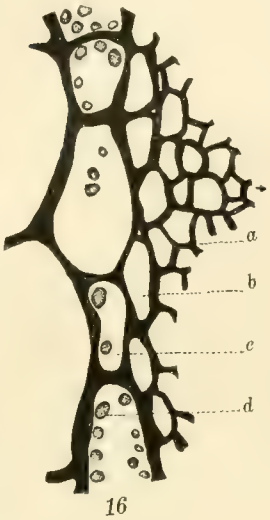

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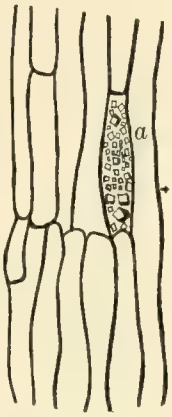

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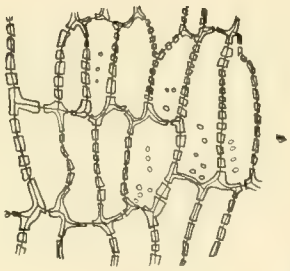

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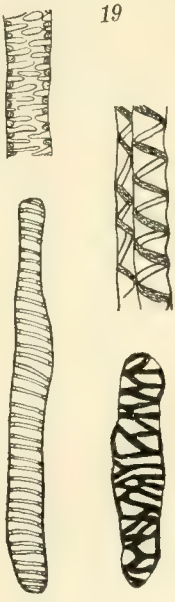

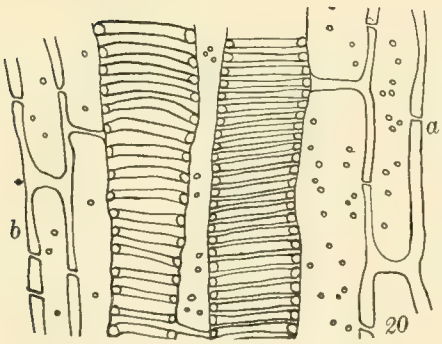

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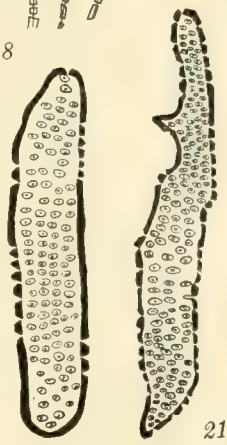


21

PLATE XL.

FIG. 22. Outline of leaf, showing frequency of calcium oxalate crystals. × 8.

FIG. 23. Outline of leaf, showing venation. × 8.

FIG. 24. Epidermis of leaf. × 182. *a*, elongated epidermis on top of leaf.

FIG. 25. Diagramatic drawing of tip of leaf. × 65. *a*, spine; *b*, bast fibers; *c*, tracheal tube; *d*, epidermis; *e*, palisade parenchyma; *f*, short chloroplast-bearing parenchyma; *g*, aqueous tissue; *h*, calcium oxalate crystals.

FIG. 26. Longitudinal section through vein. × 375.

FIG. 27. Cross section of palisade parenchyma and short chloroplast-bearing cells. × 375.

FIG. 28. Tangential section of short chloroplast-bearing cells. × 375.

FIG. 29. Water-storage aqueous tissue. *a*, plastids.

PLATE XL.

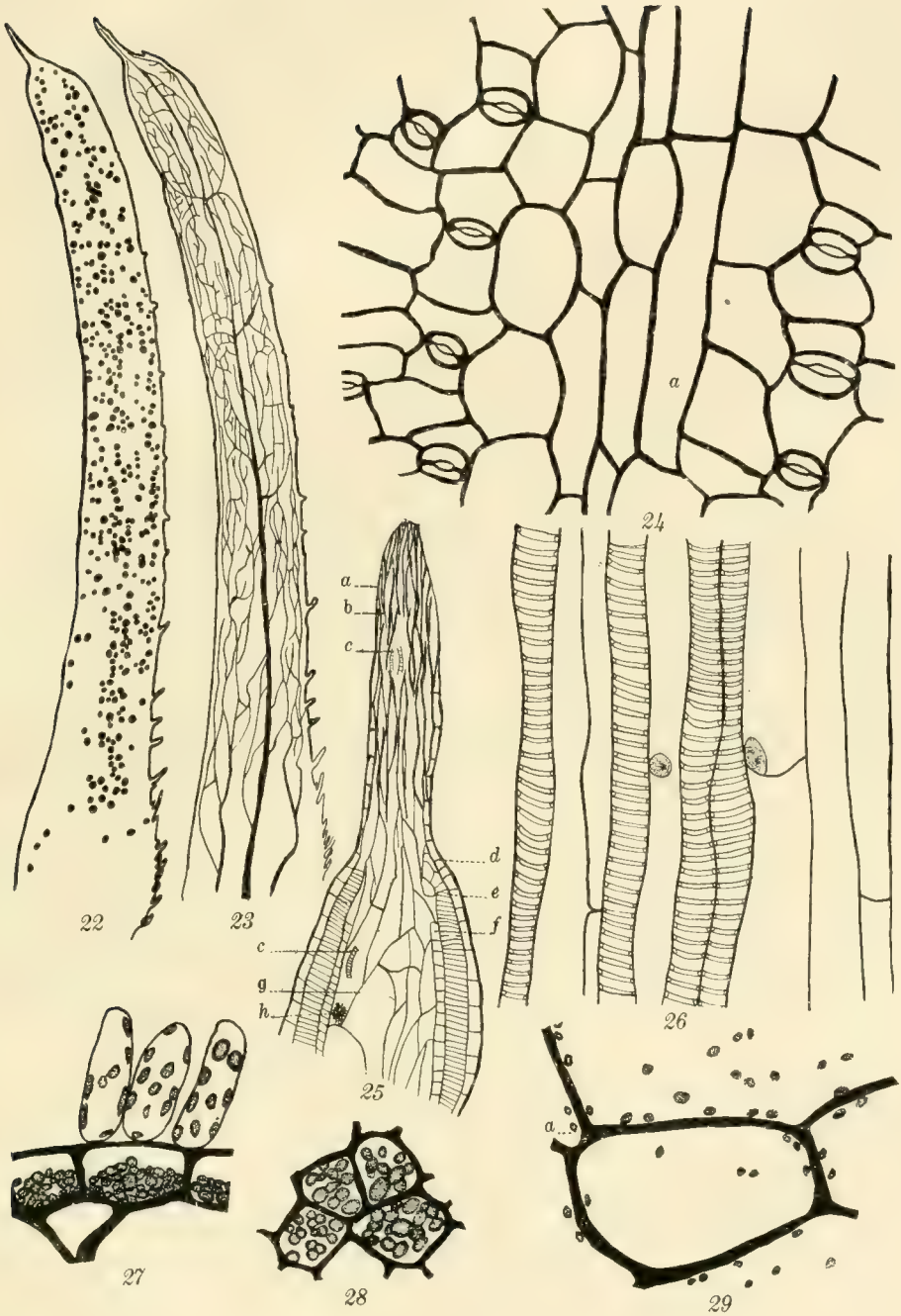


PLATE XLI.

FIG. 30. Cross section of leaf about one-third of distance from base. × 100. *a*, top; *b*, trichomes; *c*, palisade parenchyma; *d*, short chloroplast-bearing cells; *e*, branches of vein; *f*, water-storage tissue; *g*, midvein.

FIG. 31. Cross section of leaf at base of spine. × 100. *a*, bast fibers of spine.

FIG. 32. Cross section of spine. × 100.

FIG. 33. Cross section of spine. × 400.

FIG. 34. Cross section of midvein. × 570.

FIG. 35. Mascerated section of bast fibers in spine. × 250.

PLATE XLI.

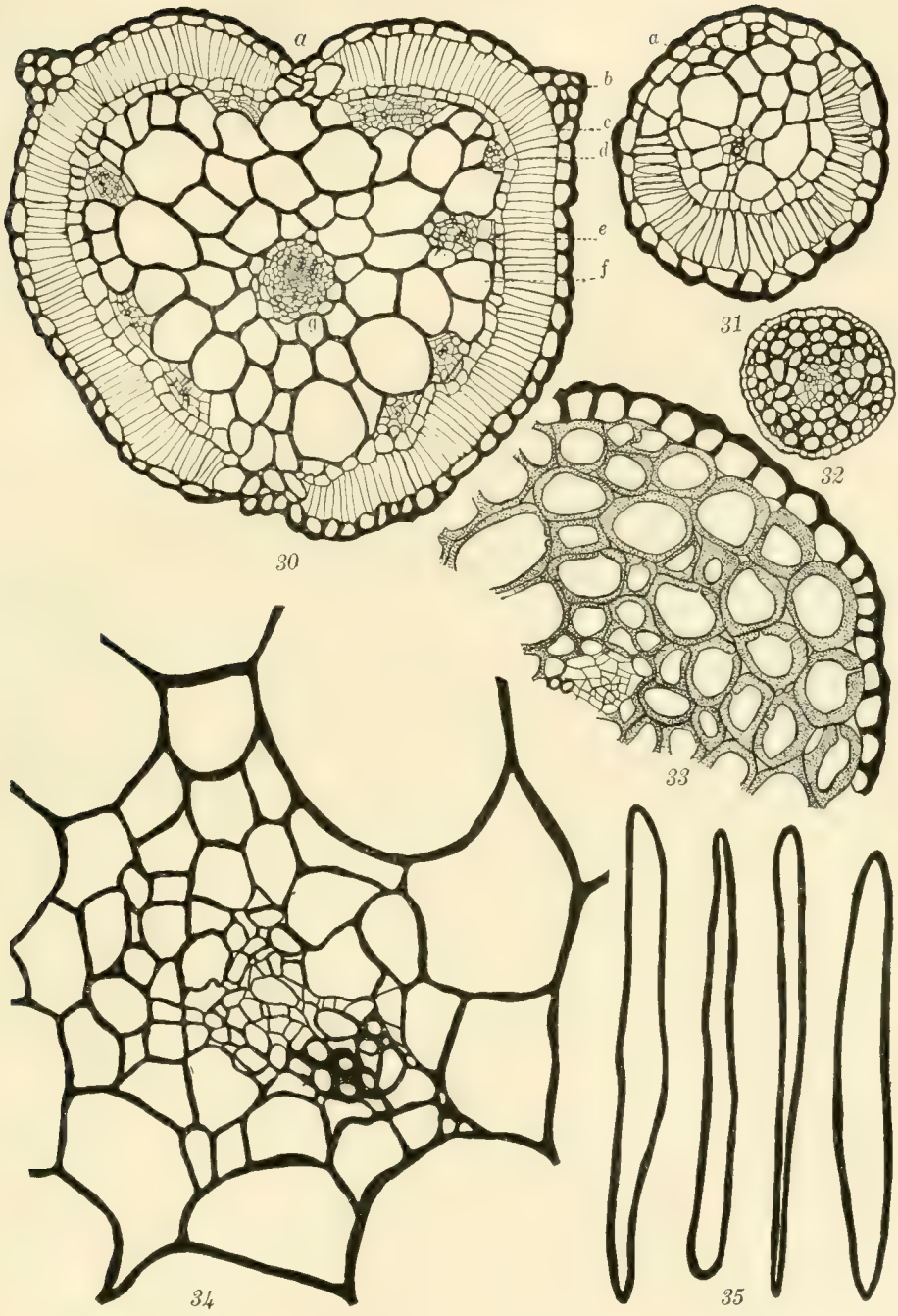


PLATE XLII.

FIG. 36. Photograph of late and early growths of stem with the spiny and succulent leaves, respectively.

PLATE XLII.

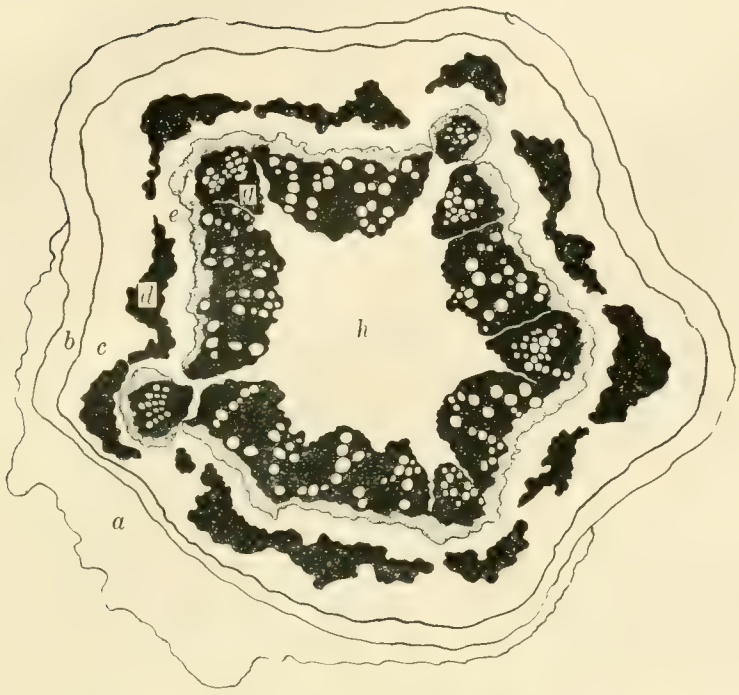
PLATE XLIII.

Gærtneria deltoidea Torr.

FIG. 1. Cross section of stem, showing at *a*, cork; *b*, collenchyma; *c*, cortex parenchyma; *d*, bast fibers; *e*, pericycle parenchyma; *f*, phloëm tissue; *g*, xylem; *h*, pith. × 100.

FIG. 2. Ditto. × 100.

PLATE XLIII.



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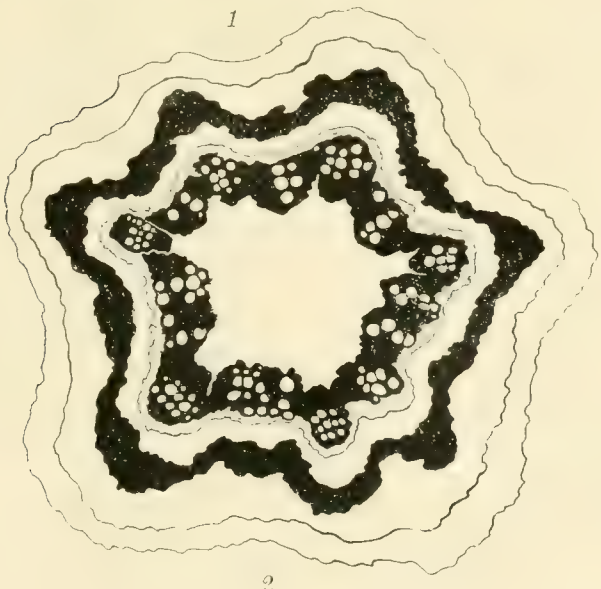

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PLATE XLIV.

Gærtneria deltoidea Torr.

FIG. 3. Cross section of stem. × 100.

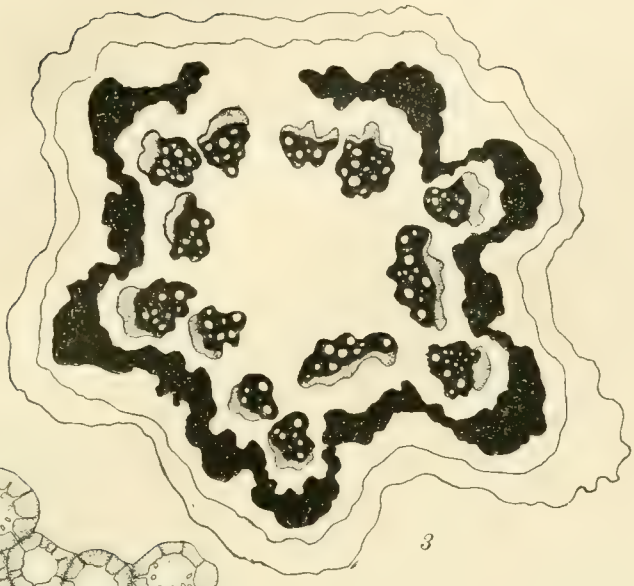
FIG. 4. Cross section of pith, old stem, lignified, showing pits. × 575.

FIG. 5. Cross section, young unlignified pith. × 620.

FIG. 6. Cross section, young pith, showing chloroplasts. × 2300.

FIG. 7. Longitudinal section, pith, lignified. × 575.

PLATE XLIV.



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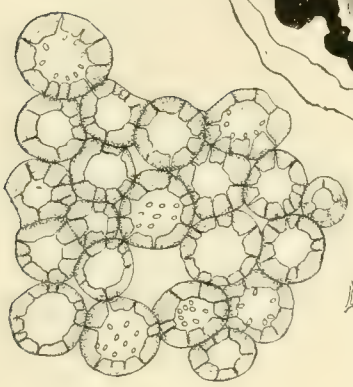

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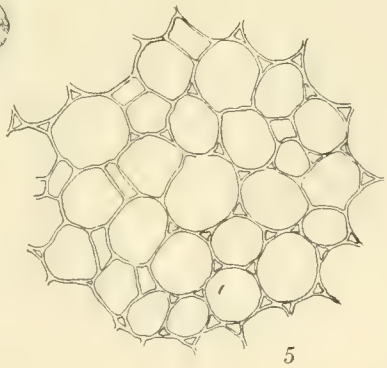

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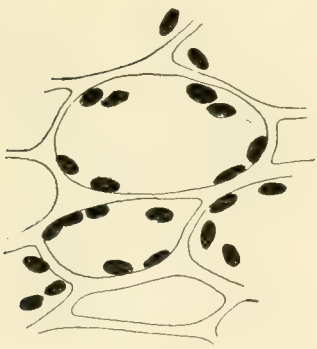

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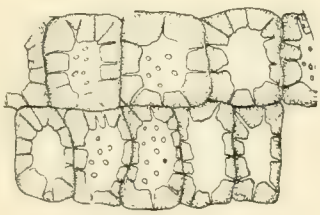


PLATE XLV.

Gærtneria deltoidea Torr.

FIG. 8. Longitudinal section, thin-walled parenchyma. × 560.

FIG. 9. Cross section, ditto, showing chloroplasts. × 560.

FIG. 10. Cross section, ditto, showing food (proteid). × 575.

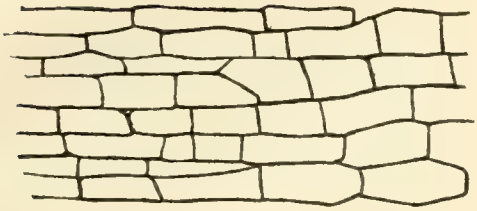
FIG. 11. Cross section, bast fibers. × 575.

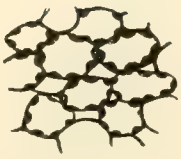
FIG. 12. Longitudinal section, stem, showing phloëm and sieve plates at *A*. × 2300.

FIG. 13. Cross section, phloëm tissue. × 575.

FIG. 14. Longitudinal section, phloëm tissue. × 575.

PLATE XLV.





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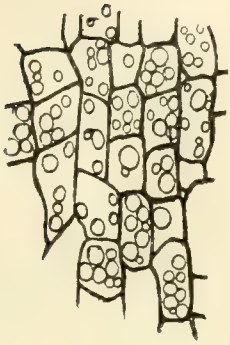

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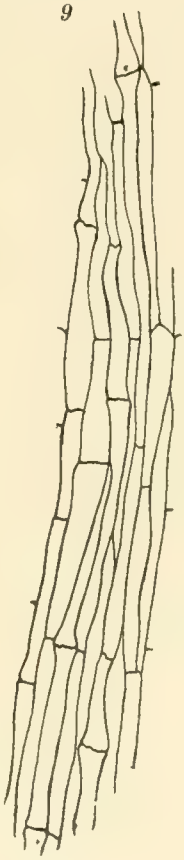

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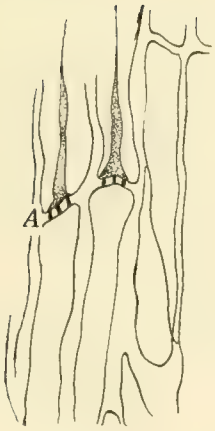

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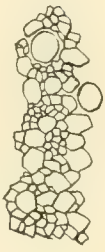

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PLATE XLVI.

Gærtneria deltoidea Torr.

FIG. 15. Showing tracheal tube elements of macerated stem. × 575.

FIG. 16. Bast and wood fibers from macerate. × 575. *a*, Wood fibers; *b*, bast fibers.

FIG. 17. Tracheal elements found in macerated stem. × 575.

FIG. 18. Longitudinal section. *A*, bast fibers; *B*, starch sheath. × 575.

FIG. 19. Cross section, xylem tissue, showing at *a*, wood fibers, and at *b*, tracheal elements. × 575.

FIG. 20. Xylem parenchyma elements found in macerate. × 575.

PLATE XLVI.

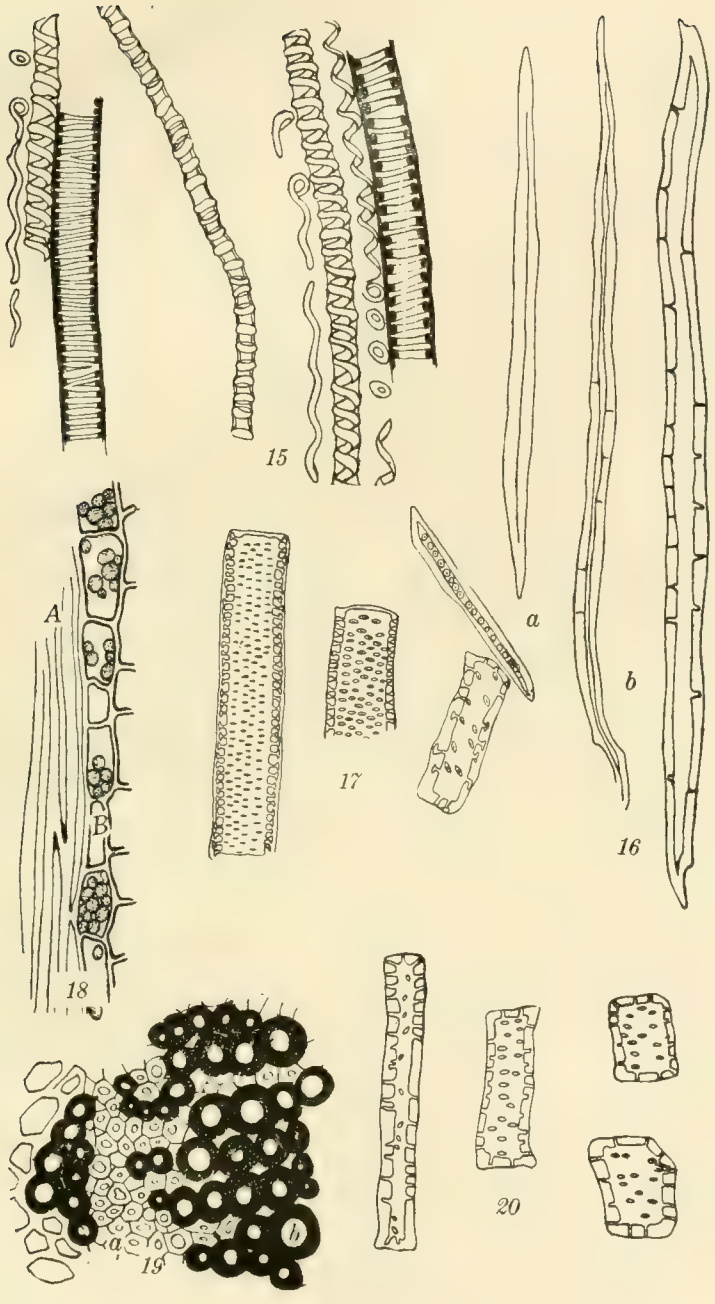


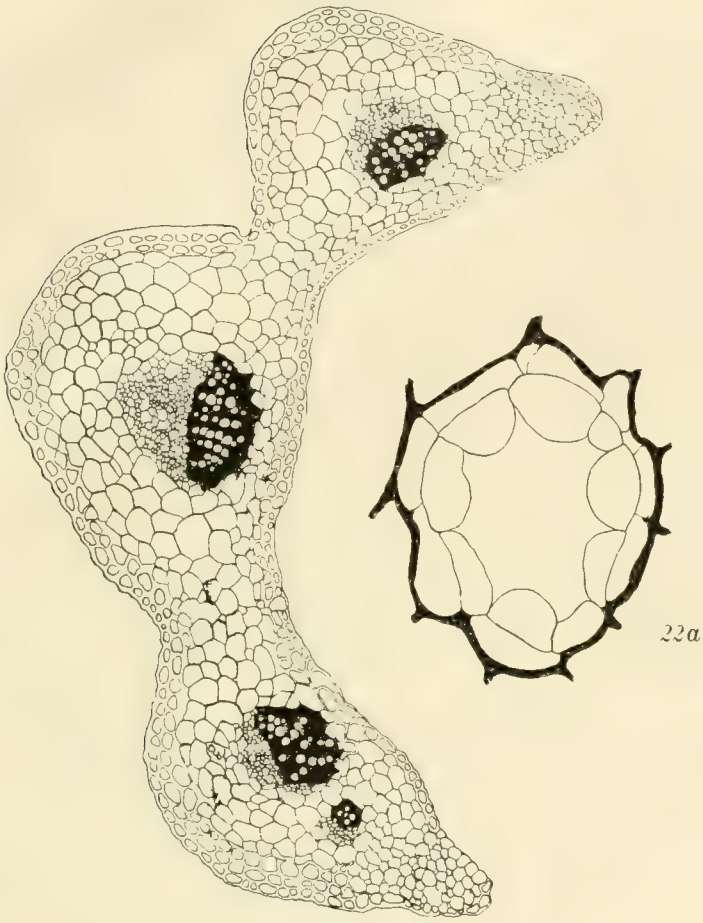
PLATE XLVII.

Gærtneria deltoidea Torr.

FIG. 21. Cross section of petiole of leaf.

FIG. 22. Cross views at *a*, *b*, and *c* of resin ducts or secreting spaces found in endodermal region of stem. × 2300.

PLATE XLVII.



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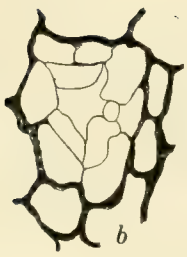
22a



b

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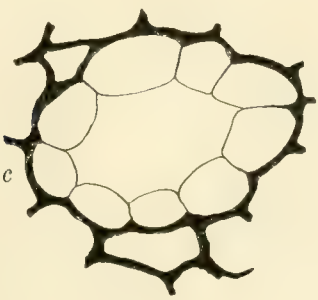

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PLATE XLVIII.

Gærtneria deltoidea Torr.

FIG. 23. Cross section, stem, from epidermis to pith region, showing at *a*, epidermis; at *b*, collenchyma; at *c*, thin-walled parenchyma; at *d*, bast fiber; at *e*, phloëm; at *f*, xylem; at *g*, pith.

FIG. 24. Cross section of epidermis of young stem, showing chloroplasts. × 2300.

FIG. 25. Longitudinal section, collenchyma tissue × 560.

FIG. 26. Cross section, collenchyma tissue × 575.

FIG. 27. Cross section, stem, showing hairs. × 560.

FIG. 28. Showing odd forms of hairs found on epidermis. × 530.

FIG. 29. Longitudinal section of stem epidermis. × 560.

PLATE XLVIII.

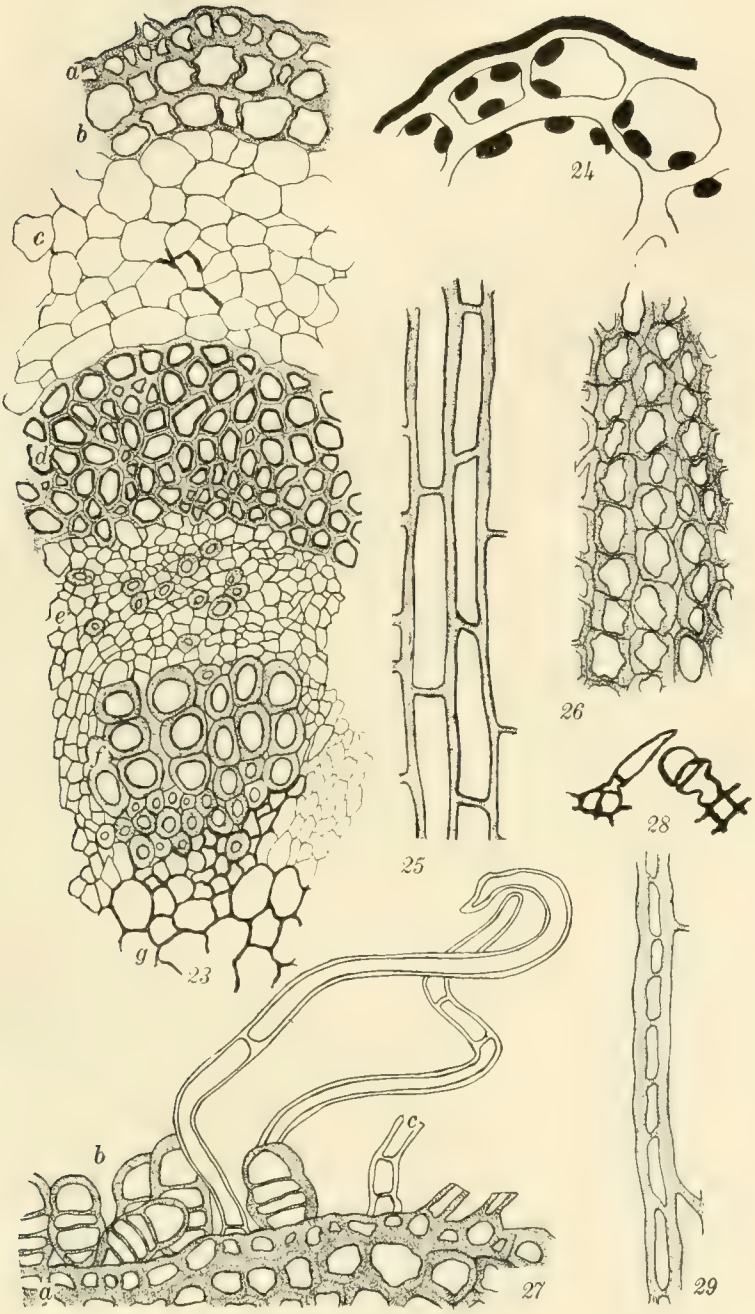


PLATE XLIX.

Gærtneria deltoidea Torr.

FIG. 30. Showing leaf venation. Drawing from bleached leaf.

FIG. 31. Cross section, leaf, showing glandular hair and stomata. × 2300.

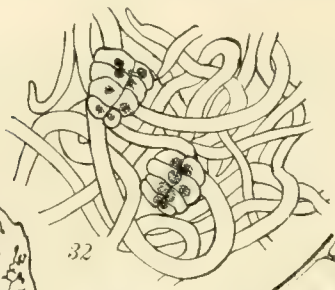
FIG. 32. View of top of leaf, showing clothing and glandular hairs. × 550.

FIG. 33. Cross section, leaf, showing attachment of clothing hairs. × 2300.

FIG. 34. *a* shows upper epidermis of leaf with stoma infected by fungus; *b* shows lower epidermis and stoma of above leaf infected by fungus.

PLATE XLIX.

31



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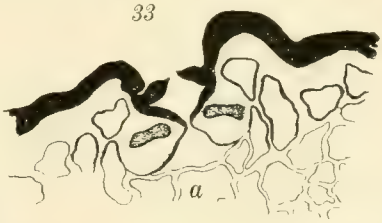


a

b

34

PLATE L.

Gærtneria deltoidea Torr.

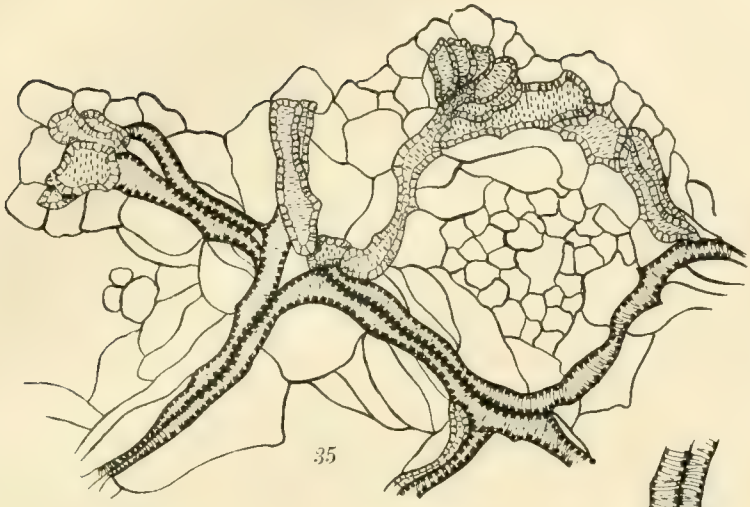
FIG. 35. Tangential section of leaf, showing tracheal elements, and adjacent parenchyma cells. × 575.

FIG. 36. Composite drawing showing epidermal cells at surface, palisade tissue just beneath, and at deepest focus the tracheal elements. × 400.

FIG. 37. Showing tangential section of tracheal elements and thick-walled water-storage parenchyma tissue. × 550.

FIG. 38. Surface view of leaf epidermis. × 560.

PLATE L.



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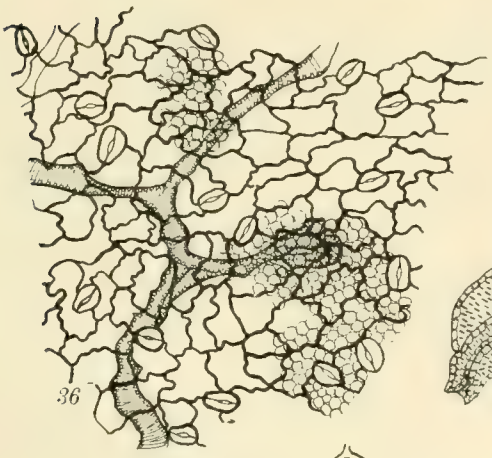

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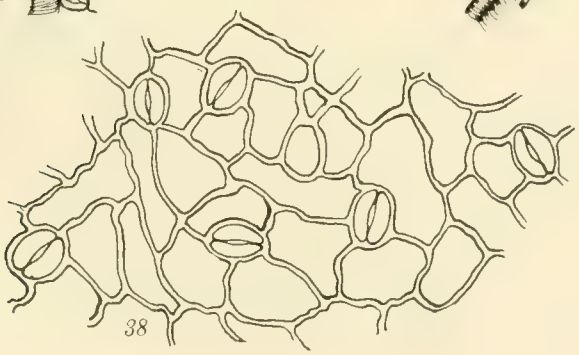

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PLATE LI.

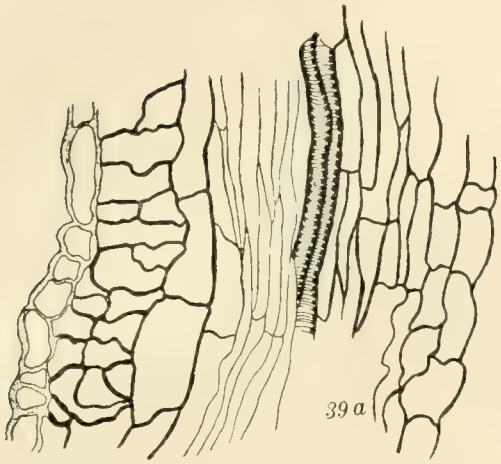
Gærtneria deltoidea Torr.

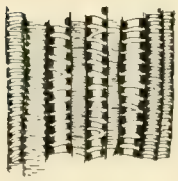
FIG. 39. Cross sections of the same leaf at different places. × 400.

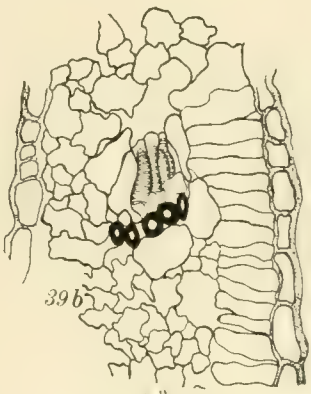
FIG. 40. Showing top view of glandular hair. × 2300.

FIG. 41. Tracheal elements in midrib. × 530.

PLATE LI.







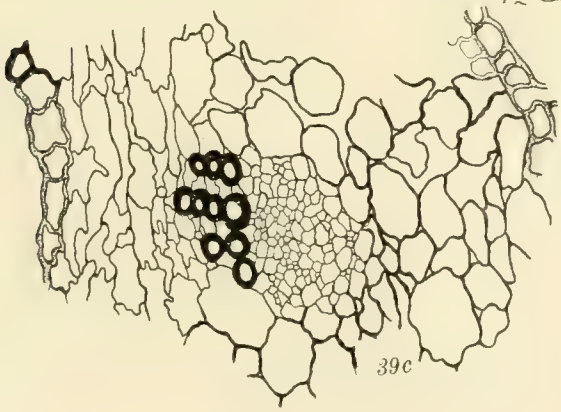


PLATE LII.

Gærtneria deltoidea Torr.

FIG. 42. Showing cross section of leaf with palisade tissue going through from one epidermis to the other. × 2300.

FIG. 43. Showing cross section of leaf with storage parenchyma cells containing chloroplasts, at *a*.

FIG. 44. Showing cross view of xylem tissue of leaf. × 2300.

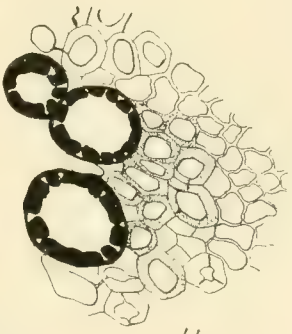
FIG. 45. Showing modified stone cells found in cross section of leaf infected by fungal growth.

FIG. 46. Cross view of leaf, showing palisade tissue, *a*, abutting on a vein; *b*, thick-walled water-storage parenchyma.

PLATE LII.

42

43



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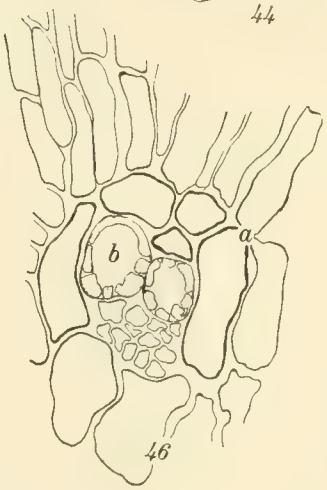

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PLATE LIII.

Gærtneria deltoidea Torr.

FIG. 47. Cross section, root, showing, *a*, root hairs; *b*, cortex tissue; *c*, pericycle; *d*, stone cells; *e*, phloëm; *f*, xylem. × 170.

FIG. 48. Showing cross section of xylem of root. × 700.

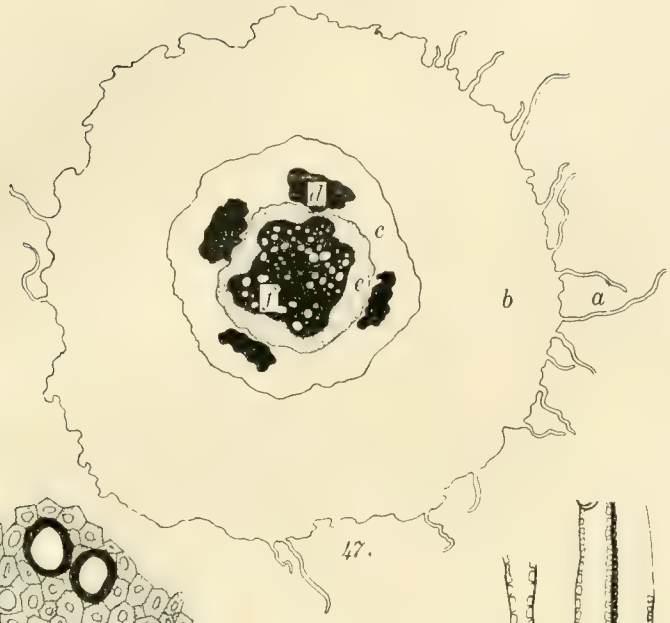
FIG. 49. Longitudinal section, root. *a*, wood parenchyma; *b*, tracheal tubes.

FIG. 50. Longitudinal section of root xylem, showing, *a*, wood fibers; *b*, tracheids; *c*, tracheal tubes. × 525.

FIG. 51. Longitudinal section of root xylem, showing ring-formed remnants of end walls of tracheal elements. × 350.

FIG. 52. Cross section of stone cells of root. × 700.

PLATE LIII.



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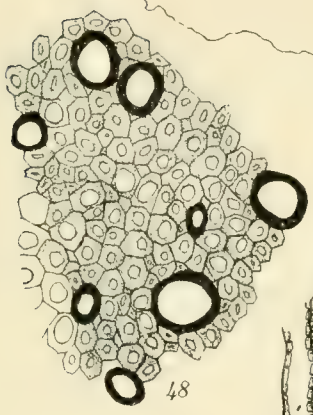

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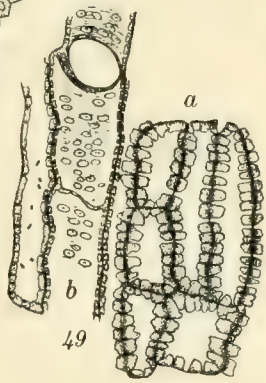


49

50 b

52

PLATE LIV.

Gærtneria deltoidea Torr.

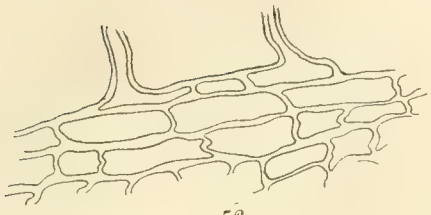
FIG. 53. Longitudinal section, root hairs and root epidermis. × 625.

FIG. 54. Cross section, ditto. × 625.

FIG. 55. Cross section of suberized cortical tissue with still functioning nuclei. × 700.

FIG. 56. Wood fibers of root. × 330.

PLATE LIV.



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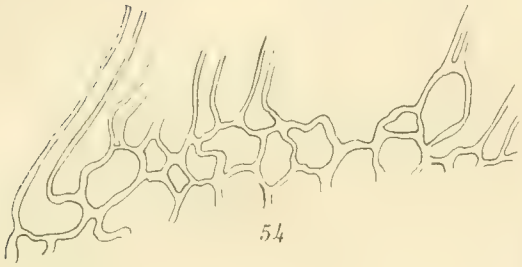

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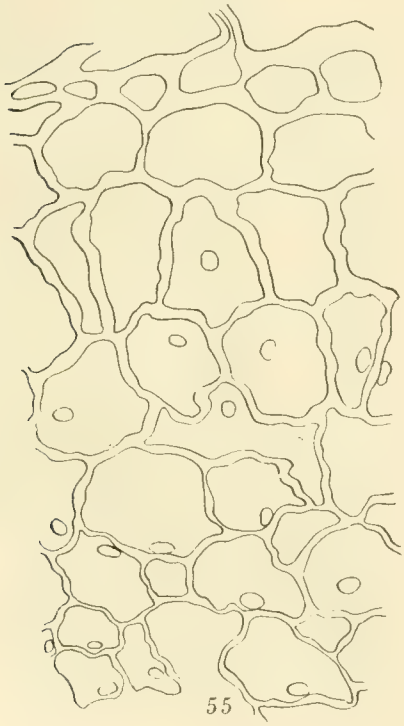

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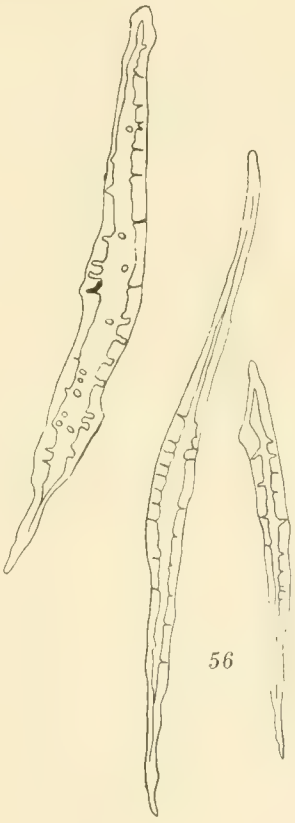

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PLATE LV.

FIG. 1. Cross section of young root. × 292. *a*, xylem; *b*, phloëm; *c*, protoxylem point; *d*, heavy-walled endodermis; *e*, cortex.

FIG. 2. Cross section of stem. × 63. *l*, pith; *q*, air space in pith; *u*, large cell in central portion of pith; *k*, xylem; *j*, phloëm; *h*, bast fiber bundle; *f*, epidermis; *g*, strengthened corner of stem; *i*, chlorophyll tissue.

FIG. 3. Stem epidermis. × 292. *g*, epidermal cell; *h*, stoma; *f*, glandular hair.

FIG. 4. Specialized glandular hair (*m*) from stem. × 292.

FIG. 5. Hairs from leaf. Similar to those on stem. × 63.

FIG. 6. Cross section of stem at corner. × 292. *a*, bast fibers; *c*, sheath cell; *v*, parenchyma inside xylem, resembling phloëm; *p*, lignified pith cells.

FIG. 7. Cross section of stem at side. × 292. *o*, chlorophyll cells; *p*, air space; *n*, secondary epidermis; *q*, phloëm; *d*, lignified parenchyma connecting bundles; *y*, wood parenchyma; *s*, water tube; *z*, pith cell.

PLATE LV.

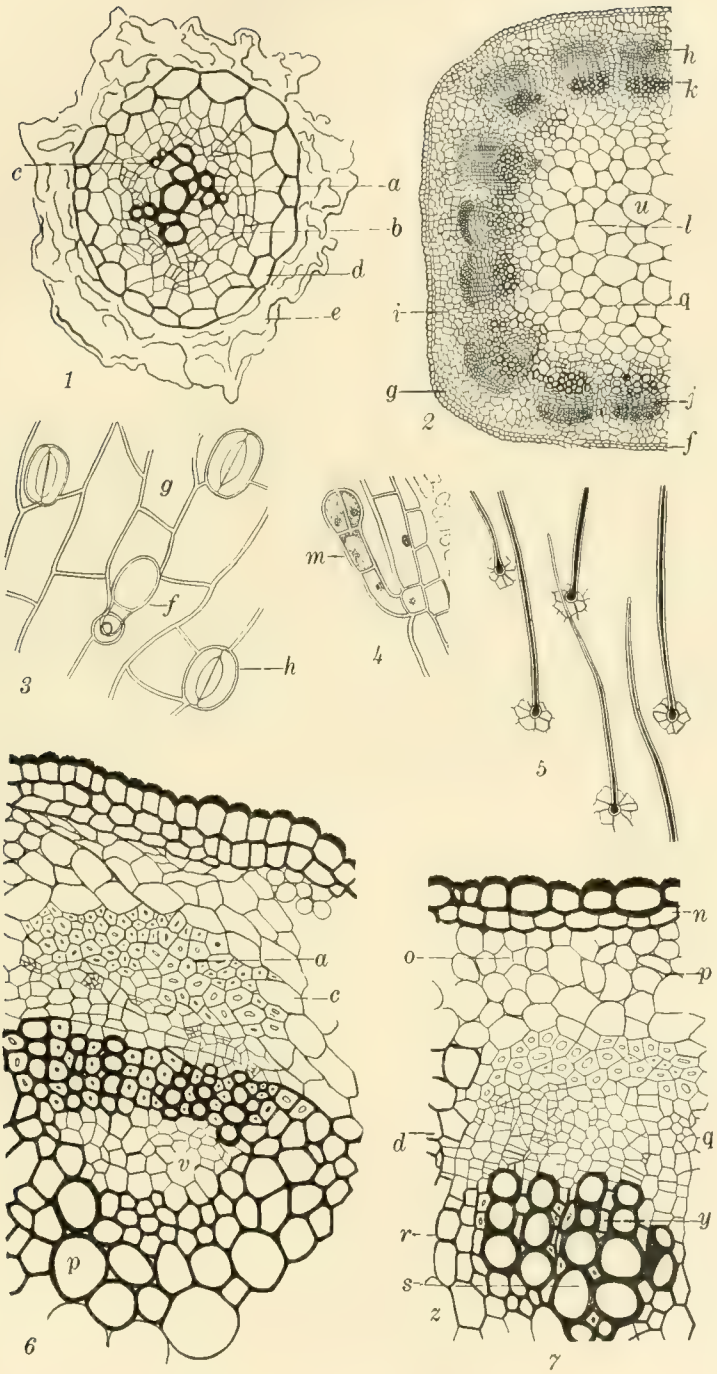


PLATE LVI.

FIG. 8. Tangential section of stem through chlorophyll region. × 618. *b*, chloroplasts; *s*, nucleus.

FIG. 9. Longitudinal section of stem. × 292. *x*, spiral tracheal tubes; *w*, row of pitted tracheids; *o*, phloëm; *i*, sclerenchyma sheath; *t*, bast fibers; *u*, epidermis; *v*, chlorophyll cells.

FIG. 10. Crystals from stone cells. × 618.

FIG. 11. Tangential section of stem through two groups of bast fibers. × 292. *w*, *j*, bast fibers; *k*, overlapping of ends of fibers; *y*, stone cells; *z*, lignified parenchyma.

FIG. 12. Bast fiber with *m*, straight pit. × 292.

FIG. 13. Special bordered pits in tracheids. *t*, pit-like structure. × 618.

FIG. 14. Epidermis of leaf. × 350.

FIG. 15. Trichome from leaf. × 350. *c*, papilla; *d*, core-like center.

FIG. 16. Stoma, cross section of leaf. × 618. *p*, air space; *q*, guard cell.

FIG. 17. Epidermis of leaf, showing cutinization by heavy line. × 292. *n*, upper surface; *o*, lower surface; *p*, lower surface under midrib.

FIG. 18. Sieve tubes from phloëm. *b*, sieve plate. × 618.

FIG. 19. Overlapping of tracheids in rows. × 618.

PLATE LVI.

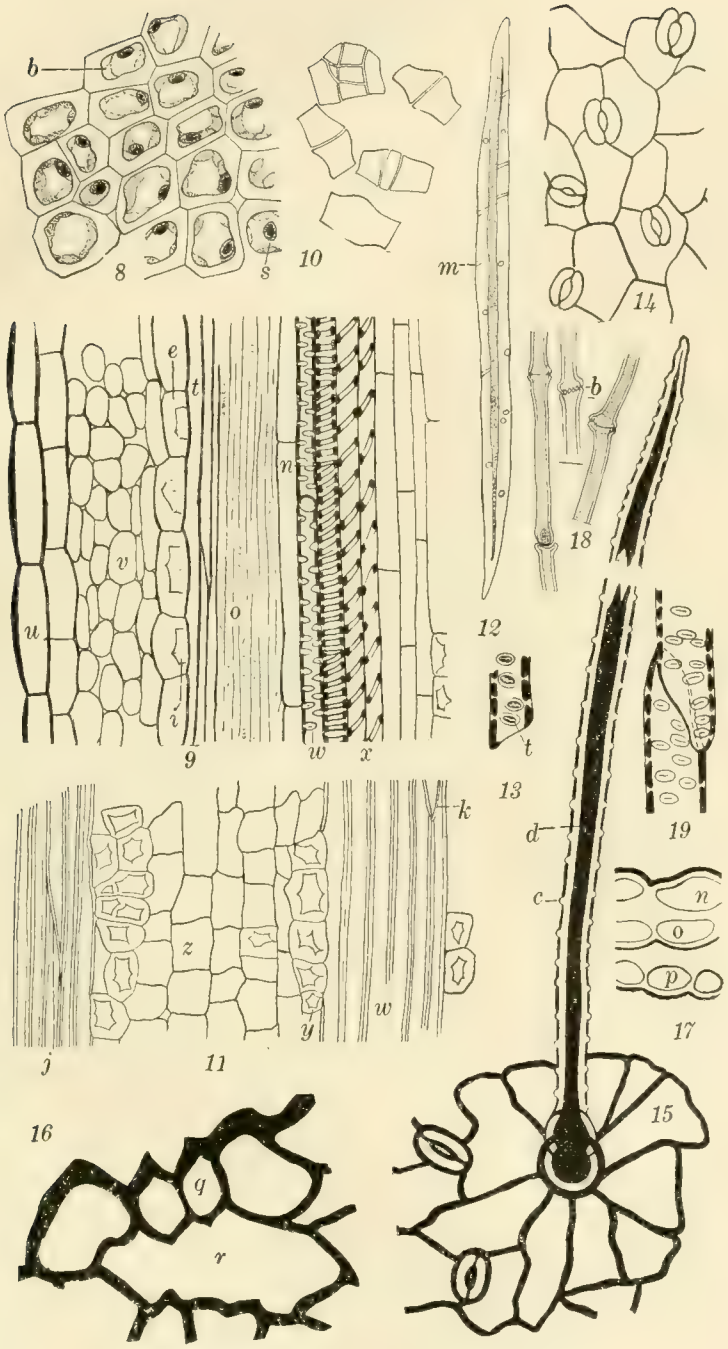


PLATE LVII.

FIG. 20. Composite drawing of bleached leaf. × 350. *b*, epidermis; *c*, stoma; *d*, trichome; *e*, palisade cells; *f*, veins.

FIG. 21. Longitudinal section through midrib of leaflet. × 292. *t*, bast fibers; *u*, wood parenchyma; *v*, tracheids; *w*, stone cells.

FIG. 22. Cross section of leaf. × 350. *k*, upper epidermis; *o*, stoma; *f*, *g*, *c*, air chamber; *d*, palisade tissue; *e*, vein; *h*, parenchyma; *b*, lower epidermis.

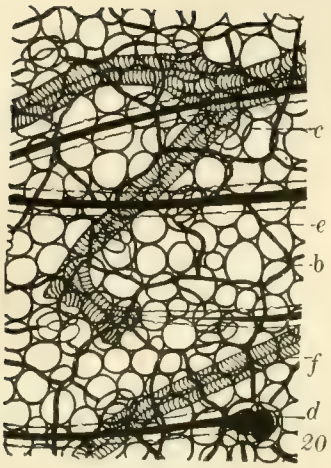
FIG. 23. From midrib of leaf. Sheath broken, showing bast fibers (*m*) beneath. × 292.

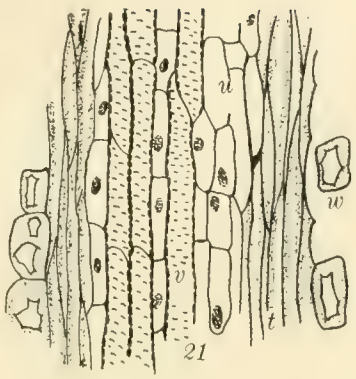
FIG. 24. Cross section of leaf petiole. (*z*) bundle. × 63.

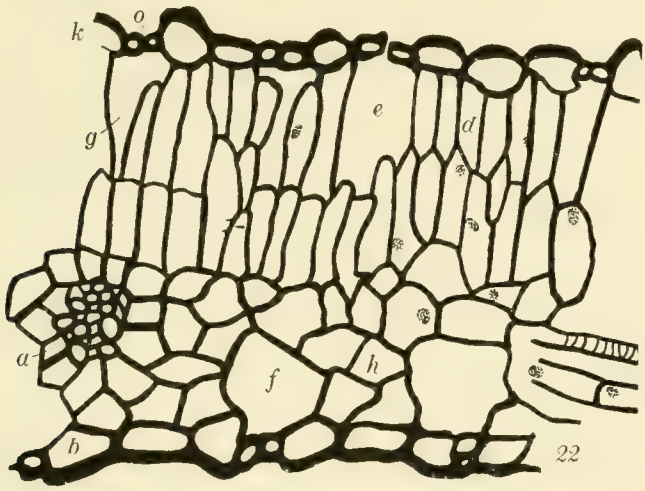
FIG. 25. Tangential section leaflet petiole. × 63.

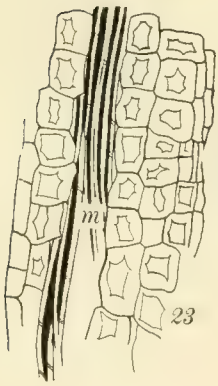
FIG. 26. Starch-containing cells from leaflet petiole. × 618.

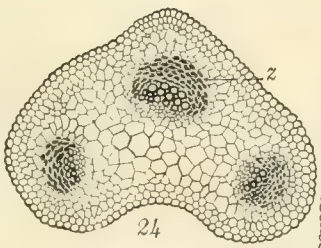
PLATE LVII.











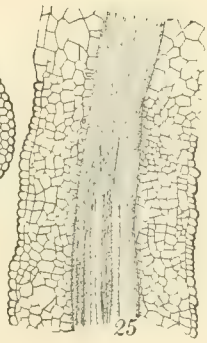


PLATE LVIII.

FIG. 27. Sheath of midrib near tip. × 292. Differentiation of border parenchyma (*l*) into stone cells (*j*).

FIG. 28. Extreme tip of midrib sheath. × 292.

FIG. 29. Drawings of leaf veins with border parenchyma. × 292.

FIG. 30. Ultimate endings of leaf veinlets in groups of spiral tracheids. × 292.

FIG. 31. Cross section of leaf base. × 63. *k*, upper epidermis; *r*, palisade tissue; *m*, lateral veins; *i*, spongy parenchyma; *j*, phloëm of midrib; *l*, xylem of midrib; *p*, lower epidermis.

FIG. 32. End of tracheal tube. × 618.

FIG. 33. Cross section of midrib from figure 31. × 292. *a*, xylem; *b*, phloëm.

FIG. 34. Bast fiber. No pits visible. × 292.

FIG. 35. Cross section of leaflet petiole. × 292. *c*, starch-containing cells; *d*, stone cells; *e*, phloëm; *f*, xylem.

FIG. 36. Water-storage tissue under midrib of leaf. × 350.

FIG. 37. Cross section of xylem in midrib. × 1233. *g*, tubes; *h*, wood parenchyma.

FIG. 38. Cells of sclerenchyma sheath. × 618.

FIG. 39. Group of tracheids from leaf. × 292.

FIG. 40. Row of tracheids from leaf. × 292.

PLATE LVIII.

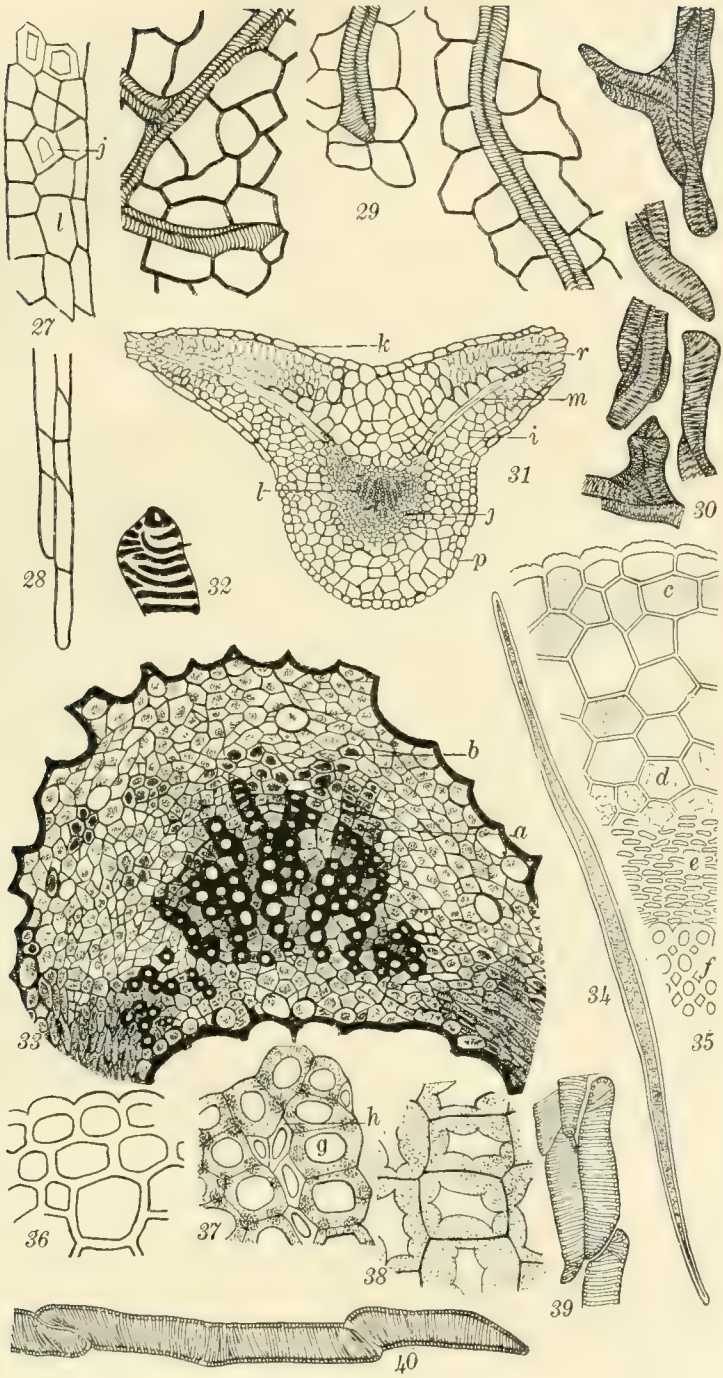


PLATE LIX.

FIG. 41. Bleached leaf showing veining system. × 15. *s*, drip tips; *z*, midrib; *y*, sheath surrounding midrib; *x*, bunch of tracheids.

PLATE LIX.

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" III. No. 1, supply exhausted. No. 2, supply exhausted. No. 3, supply small; price, 75 cents. No. 4, supply small; price, 50 cents. Price per volume, $3.
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" V. Nos. 1 and 2, supply exhausted. Price per volume, $2. Volume V consists of only two numbers.
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" VI, B. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; price per volume, $2.
" VII, A. Nos. 1, 2, 3, and 4, supply exhausted. Price per volume, $3.
" VII, B. Nos. 1 and 2 (double number), 3, and 4, supply large; price per number, 50 cents; price per volume, $2.
" VIII, A. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; price per volume, $2.
" VIII, B. No. 1, supply small; price 50 cents. Publication suspended.
" IX, A. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; per volume, $2.
" X, A. Nos. 1, 2, 3, and 4, supply large; price per number, 50 cents; per volume, $2.

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" II. Nos. 1 to 15, price, $3; supply large.
" III. Nos. 1 to 10, price, $3; supply large.
" IV. Nos. 1 to 20, price, $3; supply large.
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" IV, 1898. Upper Cretaceous Paleontology; 32 cents.
" V, 1899. Gypsum and Gypsum Cement Plasters; 16 cents.
" VI, 1900. Carboniferous Invertebrates and Cretaceous Fishes; 20 cents.
" VII, 1902. Special Report on Mineral Waters; 16 cents.
" VIII, 1904. Special Report on Lead and Zinc.
" IX, 1909. Special Report on Oil and Gas.

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Report for 1898; supply large. Postage, 8 cents.
Report for 1899; supply large. Postage, 4 cents.
Report for 1900-1901; supply small. Postage, 5 cents.
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